

# ON $C^1$ , $C^2$ , AND WEAK TYPE-(1, 1) ESTIMATES FOR LINEAR ELLIPTIC OPERATORS: PART II

HONGJIE DONG, LUIS ESCAURIAZA, AND SEICK KIM

**ABSTRACT.** We extend and improve the results in [6]: showing that weak solutions to full elliptic equations in divergence form with zero Dirichlet boundary conditions are continuously differentiable up to the boundary when the leading coefficients have Dini mean oscillation and the lower order coefficients verify certain conditions. Similar results are obtained for non-divergence form equations. We extend the weak type-(1, 1) estimates in [6] and [7] up to the boundary and derive a Harnack inequality for non-negative adjoint solutions to non-divergence form elliptic equations, when the leading coefficients have Dini mean oscillation.

## 1. INTRODUCTION AND MAIN RESULTS

Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain. We consider a second-order elliptic operator  $L$  in divergence form

$$Lu = \sum_{i,j=1}^n D_i(a^{ij}(x)D_j u) + b^i(x)u + \sum_{i=1}^n c^i(x)D_i u + d(x)u, \quad (1.1)$$

where the coefficients  $\mathbf{A} = (a^{ij})_{i,j=1}^n$ ,  $\mathbf{b} = (b^1, \dots, b^n)$ ,  $\mathbf{c} = (c^1, \dots, c^n)$ , and  $d$  are measurable functions defined on  $\overline{\Omega}$ . We assume that the principal coefficients  $\mathbf{A} = (a^{ij})$  are defined on  $\mathbb{R}^n$  and satisfy the uniform ellipticity condition

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^n a^{ij}(x)\xi^i\xi^j, \quad \forall \xi = (\xi^1, \dots, \xi^n) \in \mathbb{R}^n, \quad \forall x \in \mathbb{R}^n \quad (1.2)$$

and the uniform boundedness condition

$$\sum_{i,j=1}^n |a^{ij}(x)|^2 \leq \Lambda^2, \quad \forall x \in \mathbb{R}^n \quad (1.3)$$

for some positive constants  $\lambda$  and  $\Lambda$ .

We say that a measurable function  $\omega : (0, a) \rightarrow \mathbb{R}$  is a Dini function provided that there are constants  $c_1, c_2 > 0$  such that

$$c_1\omega(t) \leq \omega(s) \leq c_2\omega(t) \quad (1.4)$$

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whenever  $\frac{1}{2}t \leq s \leq t$  and  $0 < t < a$  and that

$$\int_0^t \frac{\omega(s)}{s} ds < +\infty, \quad \forall t \in (0, a).$$

For  $x \in \mathbb{R}^n$  and  $r > 0$ , we denote by  $B(x, r)$  the Euclidean ball with radius  $r$  centered at  $x$ , and denote

$$\Omega(x, r) := \Omega \cap B(x, r).$$

For a locally integrable function  $g$  on  $\Omega$ , we shall say that  $g$  is uniformly Dini continuous (in  $\Omega$ ) if the function  $\varrho_g : \mathbb{R}_+ \rightarrow \mathbb{R}$  defined by

$$\varrho_g(r) := \sup_{x \in \Omega} \sup_{y, y' \in \Omega(x, r)} |g(y) - g(y')|$$

is a Dini function, while we shall say that  $g$  is of *Dini mean oscillation* (in  $\Omega$ ) if the function  $\omega_g : \mathbb{R}_+ \rightarrow \mathbb{R}$  defined by

$$\omega_g(r) := \sup_{x \in \overline{\Omega}} \int_{\Omega(x, r)} |g(y) - \bar{g}_{\Omega(x, r)}| dy \quad \left( \bar{g}_{\Omega(x, r)} := \int_{\Omega(x, r)} g \right)$$

is a Dini function. We point out that the condition (1.4) is satisfied by  $\varrho_g$  and also by  $\omega_g$ ; see [14]. Moreover, it should be clear that if  $g$  is uniformly Dini continuous, then it is of Dini mean oscillation and  $\omega_g(r) \leq \varrho_g(r)$ . It is worthwhile to note that if  $\Omega$  is such that

$$|\Omega(x, r)| \geq A_0 r^n, \quad 0 < \forall r \leq \text{diam } \Omega \quad (A_0 \text{ is a positive constant}) \quad (1.5)$$

and if  $g$  is of Dini mean oscillation, then  $g$  is uniformly continuous with a modulus of continuity determined by  $\omega_g$ .

In a recent paper [14], Yanyan Li raised a question whether weak solutions of

$$\sum_{i,j=1}^n D_i(a^{ij}(x)D_j u) = 0$$

are continuously differentiable when  $\mathbf{A} = (a^{ij})$  are of Dini mean oscillation.<sup>1</sup> In [6], the first and third named authors showed that the answer to his question is positive. This paper is a sequel to [6] and extends and improves results presented there. More precisely, we show that weak solutions to (1.1) with zero Dirichlet boundary conditions are continuously differentiable up to boundary provided that the leading coefficients  $\mathbf{A}$  and  $\mathbf{b}$  are of Dini mean oscillation, lower order coefficients  $c$  and  $d$  belong to  $L^q$  with  $q > n$ , and  $\partial\Omega$  has  $C^{1,Dini}$  boundary. We prove a similar result when the operator is in non-divergence form. In [7], the second named author investigated (interior) weak type-(1, 1) estimates for solutions of

$$\sum_{i,j=1}^n a^{ij}(x)D_{ij}u = f \text{ in } B(0, 1), \quad u = 0 \text{ on } \partial B(0, 1),$$

and showed that if  $\mathbf{A} = (a^{ij})$  belong to the class of functions with vanishing mean oscillations (VMO), then the  $D^2u$  satisfies weak type-(1, 1) estimates with respect to  $W dx$ . Here  $W$  is a nonnegative solution to the adjoint equation, which is a good Muckenhoupt weight as  $\log W$  was proved to be in VMO, so that the associated measure  $W dx$  is better adjusted to the equation than  $dx$ . Moreover, it is also shown

<sup>1</sup>In fact, the condition on  $\mathbf{A}$  imposed by Yanyan Li was slightly stronger.

in [7] that the standard weak type-(1, 1) estimates (i.e., the estimate with  $W = 1$ ) do not hold even if  $\mathbf{A}$  is uniformly continuous. In this paper, we prove that if  $\mathbf{A}$  is of Dini mean oscillation, then the standard weak type-(1, 1) estimates hold up to the boundary. We also show that in this case, the weight  $W$  mentioned above satisfies a Harnack type inequality.

Now, we state the main results more precisely.

**Definition 1.6.** Let  $\Omega \subset \mathbb{R}^n$  be open and bounded,  $k = 1, 2, \dots$ . We say  $\partial\Omega$  is  $C^{k,Dini}$  if for each point  $x_0 \in \partial\Omega$ , there exist  $r > 0$  independent of  $x_0$  and a  $C^{k,Dini}$  function (i.e.,  $C^k$  function whose  $k$ th derivatives are Dini continuous)  $\gamma : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$  such that (upon relabeling and reorienting the coordinates axes if necessary) in a new coordinate system  $(x', x^n) = (x^1, \dots, x^{n-1}, x^n)$ ,  $x_0$  becomes the origin and

$$\Omega \cap B(0, r) = \{x \in B(0, r) : x^n > \gamma(x^1, \dots, x^{n-1})\}, \quad \gamma(0') = 0.$$

**Condition 1.7.** The coefficients  $\mathbf{A} = (a^{ij})_{i,j=1}^n$  and  $\mathbf{b} = (b^1, \dots, b^n)$  are of Dini mean oscillation in  $\Omega$  and  $\mathbf{c} = (c^1, \dots, c^n)$ ,  $d \in L^q(\Omega)$  with  $q > n$ .

**Theorem 1.8.** Let  $\Omega$  have  $C^{1,Dini}$  boundary and the coefficients of  $L$  in (1.1) satisfy the conditions (1.2), (1.3), and Condition 1.7. Let  $u \in W_0^{1,2}(\Omega)$  be the weak solution of

$$Lu = \operatorname{div} \mathbf{g} + f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where  $\mathbf{g} = (g^1, \dots, g^n)$  are of Dini mean oscillation in  $\Omega$  and  $f \in L^q(\Omega)$  with  $q > n$ . Then, we have  $u \in C^1(\overline{\Omega})$ .

The proof of Theorem 1.8 is given Section 2, where an upper bound for the modulus of continuity of  $Du$  can be found.

We also consider elliptic operators  $\mathcal{L}$  in non-divergence form

$$\mathcal{L}u = \sum_{i,j=1}^n a^{ij}(x) D_{ij}u + \sum_{i=1}^n b^i(x) D_iu + c(x)u, \quad (1.9)$$

where the coefficients  $\mathbf{A}$  are assumed to be symmetric, i.e.  $a^{ij} = a^{ji}$ , and satisfy the uniform ellipticity and boundedness condition

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^n a^{ij}(x) \xi^i \xi^j \leq \Lambda|\xi|^2, \quad \forall \xi = (\xi^1, \dots, \xi^n) \in \mathbb{R}^n, \quad \forall x \in \mathbb{R}^n \quad (1.10)$$

for some constants  $0 < \lambda \leq \Lambda$ .

**Condition 1.11.** The coefficients  $\mathbf{A} = (a^{ij})_{i,j=1}^n$ ,  $\mathbf{b} = (b^1, \dots, b^n)$ , and  $c$  are of Dini mean oscillation in  $\Omega$ .

**Theorem 1.12.** Let  $\Omega$  have  $C^{2,Dini}$  boundary and the coefficients of  $\mathcal{L}$  in (1.9) satisfy the condition (1.10) and Condition 1.11. Let  $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$  be a strong solution of

$$\mathcal{L}u = g \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where  $g$  is of Dini mean oscillation in  $\Omega$ . Then, we have  $u \in C^2(\overline{\Omega})$ .

The proof of Theorem 1.12 is also given Section 2. The formal adjoint operator of the non-divergence operator  $\mathcal{L}$  is defined by

$$\mathcal{L}^*u = D_{ij}(a^{ij}(x)u) - D_i(b^i(x)u) + c(x)u.$$

We also deal with the boundary value problem

$$\mathcal{L}^* u = \operatorname{div}^2 \mathbf{g} + f \text{ in } \Omega, \quad u = \psi + \frac{\mathbf{g}^v \cdot v}{\mathbf{A}v \cdot v} \text{ on } \partial\Omega, \quad (1.13)$$

where  $\mathbf{g} = (g^{kl})_{k,l=1}^n$  is a symmetric matrix and  $\operatorname{div}^2 \mathbf{g} = \sum_{k,l=1}^n D_{kl} g^{kl}$ . At first, the appearance of the term  $\frac{\mathbf{g}^v \cdot v}{\mathbf{A}v \cdot v}$  as a part of boundary value may look strange, but it helps to make  $\mathbf{g}$  to disappear from the boundary integral in the identity (1.15), which formally defines a “weak” adjoint solution to (1.13); see [9, Definition 2] for more details.

**Definition 1.14.** Let  $\Omega \subset \mathbb{R}^n$  be a bounded  $C^{1,1}$  domain with unit exterior normal vector  $v$ . Assume that  $\mathbf{g} \in L^p(\Omega)$ ,  $f \in L^p(\Omega)$ , and  $\psi \in L^p(\partial\Omega)$ , where  $1 < p < \infty$ . We say that  $u \in L^p(\Omega)$  is an adjoint solution to (1.13) if  $u$  satisfies

$$\int_{\Omega} u \mathcal{L}v \, dx = \int_{\Omega} \operatorname{tr}(\mathbf{g} D^2 v) \, dx + \int_{\Omega} f v \, dx + \int_{\partial\Omega} \psi \mathbf{A} Dv \cdot v \, dS_x, \quad (1.15)$$

for any  $v \in W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega)$ , where  $\frac{1}{p} + \frac{1}{p'} = 1$ . By a local adjoint solution of (1.13), we mean a solution in  $L_{loc}^p(\Omega)$  that verifies (1.15) when  $v$  is in  $W_0^{2,p'}(\Omega)$ .

**Condition 1.16.** The coefficients  $\mathbf{A} = (a^{ij})_{i,j=1}^n$  are of Dini mean oscillation over an open set containing  $\overline{\Omega}$  and  $\mathbf{b} = (b^1, \dots, b^n) \in L^q(\Omega)$ ,  $c \in L^{\frac{q}{2}}(\Omega)$ , for some  $q > n$ .

**Theorem 1.17.** Let  $\Omega$  have a  $C^{1,1}$  boundary, the coefficients of  $\mathcal{L}$  in (1.9) satisfy the condition (1.10) and Condition 1.16. Let  $u \in \tilde{L}^2(\Omega)$  be an adjoint solution of the problem

$$\mathcal{L}^* u = \operatorname{div}^2 \mathbf{g} + f \text{ in } \Omega, \quad u = \psi + \frac{\mathbf{g}^v \cdot v}{\mathbf{A}v \cdot v} \text{ on } \partial\Omega,$$

where  $\mathbf{g}$  is of Dini mean oscillation in  $\Omega$ ,  $f \in L^q(\Omega)$  with  $q > \frac{n}{2}$ , and  $\psi \in C(\partial\Omega)$ . Then,  $u \in C(\overline{\Omega})$ .

The proof of Theorem 1.17 is also given in Section 2. We note that in Theorem 1.17, we assume  $\mathbf{A}$  and  $\mathbf{g}$  are of Dini mean oscillation, and thus  $\frac{\mathbf{g}^v \cdot v}{\mathbf{A}v \cdot v}$  becomes a uniformly continuous function in  $\overline{\Omega}$ . Therefore, the boundary data in the above theorem include all continuous functions defined on  $\partial\Omega$ . See [15] for previous results on interior  $C^\alpha$ -regularity,  $0 < \alpha < 1$ , for solutions to (1.13) with  $\mathbf{g} = 0$ .

In section 3, we provide an improvement of the weak type-(1, 1) estimates given in [6]. In particular, they are shown to hold up to the boundary, while in the non-divergence case, the weak type-(1, 1) estimate is shown to hold without imposing further conditions on the principal coefficients  $\mathbf{A}$  other than being of Dini mean oscillation over an open set containing  $\overline{\Omega}$ .

**Theorem 1.18.** Let  $\Omega$  have a  $C^{1,Dini}$  boundary and the coefficients  $\mathbf{A} = (a^{ij})$  satisfy the conditions (1.2), (1.3), and the following:

$$\exists c > 0 \text{ such that } \omega_{\mathbf{A}}(r) \leq c(\ln r)^{-2}, \quad \forall r \in (0, \frac{1}{2}). \quad (1.19)$$

For  $\mathbf{f} = (f^1, \dots, f^n) \in L^2(\Omega)$ , let  $u \in W_0^{1,2}(\Omega)$  be a unique weak solution to

$$\sum_{i,j=1}^n D_i(a^{ij} D_j u) = \operatorname{div} \mathbf{f} \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Then for any  $t > 0$ , we have

$$|\{x \in \Omega : |Du(x)| > t\}| \leq \frac{C}{t} \int_{\Omega} |f|,$$

where  $C = C(n, \lambda, \Lambda, \omega_{\mathbf{A}}, \Omega)$ .

A similar result can be proved for the adjoint problem

$$\sum_{i,j=1}^n D_{ij}(a^{ij}u) = \operatorname{div}^2 \mathbf{g} \text{ in } \Omega, \quad u = \frac{\mathbf{g}^{\nu} \cdot \nu}{\mathbf{A} \nu \cdot \nu} \text{ on } \partial\Omega.$$

The statement and its proof are similar to those of Theorem 1.18 and omitted.

**Theorem 1.20.** *Let  $\Omega$  have a  $C^{1,1}$  boundary, the coefficients  $\mathbf{A} = (a^{ij})$  have Dini mean oscillations over an open set containing  $\overline{\Omega}$  and satisfy the condition (1.10). For  $f \in L^2(\Omega)$ , let  $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$  be the unique solution to*

$$\sum_{i,j=1}^n a^{ij} D_{ij}u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \quad (1.21)$$

Then for any  $t > 0$ , we have

$$|\{x \in \Omega : |D^2u(x)| > t\}| \leq \frac{C}{t} \int_{\Omega} |f| dx, \quad (1.22)$$

where  $C = C(n, \lambda, \Lambda, \omega_{\mathbf{A}}, \Omega)$ .

We recall in Remark 3.19 the previously known interior weak type-(1, 1) properties for solutions to (1.21) and sketch out how to extend them up to the boundary, when the leading coefficients matrix  $\mathbf{A}$  is only in VMO over an open set containing  $\overline{\Omega}$ . We also explain why Theorem 1.20 is optimal for its comparison with counterexamples in [7, §3].

The paper is organized as follows. In Section 2 we provide some preliminary lemmas and propositions and the proofs of Theorems 1.8, 1.12, and 1.17. Section 3 is devoted to the proof of Theorems 1.18 and 1.20. Section 4 is an appendix where we outline for the readers convenience a complete proof of Lemma 4.1, which is standard, and a proof of Lemma 4.9, where a Harnack type inequality for nonnegative adjoint solutions are presented. Lemma 4.1 is used in the proofs of Theorems 1.18, 1.20 and of Lemmas 2.3, 2.4, and 2.5.

Finally, a few remarks are in order. Theorems 1.8, 1.12, and 1.18 are easily extended to elliptic systems since their proofs do not use any scalar structure. The same is true for Theorem 1.17 if we keep  $\psi \equiv 0$  there. In Theorem 1.8 (resp. Theorem 1.12), instead of assuming zero Dirichlet data, we may assume  $u = \psi$  on  $\partial\Omega$  with  $\psi \in C^{1,Dini}(\overline{\Omega})$  (resp.  $\psi \in C^{2,Dini}(\overline{\Omega})$ ). Finally, the conditions on lower order terms in Theorems 1.8, and 1.17 can be relaxed a little. For example, in Theorem 1.8 we may assume that  $c$ ,  $d$ , and  $f$  belong to suitable Morrey-Campanato spaces instead of  $L^p$  spaces.

## 2. PROOF OF THEOREMS 1.8, 1.12, AND 1.17

We write  $x = (x^1, \dots, x^n) = (x', x^n)$ . Hereafter, we shall denote

$$B^+(0, r) = B(0, r) \cap \{x^n > 0\} \quad \text{and} \quad T(0, r) = B(0, r) \cap \{x^n = 0\}.$$

We fix a smooth (convex) domain  $\mathcal{D}$  satisfying  $B^+(0, \frac{1}{2}) \subset \mathcal{D} \subset B^+(0, 1)$  so that  $\partial\mathcal{D}$  contains a flat portion  $T(0, \frac{1}{2})$ . For  $\bar{x} \in \partial\mathbb{R}_+^n = \{x^n = 0\}$ , we then set

$$B^+(\bar{x}, r) = B^+(0, r) + \bar{x}, \quad T(\bar{x}, r) = T(0, r) + \bar{x}, \quad \text{and} \quad \mathcal{D}(\bar{x}, r) = r\mathcal{D} + \bar{x}.$$

Throughout the rest of paper, we adopt the usual summation convention over repeated indices. Also, for nonnegative (variable) quantities  $A$  and  $B$ , the relation  $A \lesssim B$  should be understood that there is some constant  $c > 0$  such that  $A \leq cB$ . We write  $A \approx B$  if  $A \lesssim B$  and  $B \lesssim A$ .

### 2.1. Preliminary lemmas.

**Lemma 2.1.** *Let  $\Omega$  satisfy the condition (1.5). If  $f$  is uniformly Dini continuous and  $g$  is of Dini mean oscillation in  $\Omega$ , then  $fg$  is of Dini mean oscillation in  $\Omega$ .*

*Proof.* For any  $x \in \bar{\Omega}$  and  $r > 0$ , we have

$$\begin{aligned} \int_{\Omega(x,r)} |fg - \overline{fg}_{\Omega(x,r)}| &\leq \int_{\Omega(x,r)} |fg - f \bar{g}_{\Omega(x,r)}| + \int_{\Omega(x,r)} |f \bar{g}_{\Omega(x,r)} - \overline{fg}_{\Omega(x,r)}| \\ &\leq \sup_{\Omega(x,r)} f \cdot \omega_g(r) + \varrho_f(r) \cdot \int_{\Omega(x,r)} |g|, \end{aligned}$$

where we used

$$\sup_{\Omega(x,r)} |f \bar{g}_{\Omega(x,r)} - \overline{fg}_{\Omega(x,r)}| \leq \varrho_f(r) \cdot \int_{\Omega(x,r)} |g|.$$

Therefore, we get

$$\omega_{fg}(r) \leq \|f\|_{L^\infty(\Omega)} \omega_g(r) + \|g\|_{L^\infty(\Omega)} \varrho_f(r) \quad (2.2)$$

and thus  $\omega_{fg}$  is a Dini function.  $\blacksquare$

**Lemma 2.3.** *Let  $\bar{\mathbf{A}} = (\bar{a}^{ij})$  be a constant matrix satisfying (1.2) and (1.3). For  $f \in L^2(\mathcal{D})$  let  $u \in W_0^{1,2}(\mathcal{D})$  be a unique weak solution to*

$$\sum_{i,j=1}^n D_i(\bar{a}^{ij} D_j u) = \operatorname{div} f \text{ in } \mathcal{D}; \quad u = 0 \text{ on } \partial\mathcal{D}.$$

*Then for any  $t > 0$ , we have*

$$|\{x \in \mathcal{D} : |Du(x)| > t\}| \leq \frac{C}{t} \int_{\mathcal{D}} |f|,$$

*where  $C = C(n, \lambda, \Lambda, \mathcal{D})$ .*

*Proof.* See proof of [6, Lemma 2.2] and Lemma 4.1.  $\blacksquare$

**Lemma 2.4.** *Let  $\bar{\mathbf{A}} = (\bar{a}^{ij})$  be a constant symmetric matrix satisfying (1.10). For  $f \in L^2(\mathcal{D})$  let  $u \in W^{2,2}(\mathcal{D}) \cap W_0^{1,2}(\mathcal{D})$  be a unique solution to*

$$\sum_{i,j=1}^n \bar{a}^{ij} D_{ij} u = f \text{ in } \mathcal{D}; \quad u = 0 \text{ on } \partial\mathcal{D}.$$

*Then for any  $t > 0$ , we have*

$$|\{x \in \mathcal{D} : |D^2 u(x)| > t\}| \leq \frac{C}{t} \int_{\mathcal{D}} |f|,$$

*where  $C = C(n, \lambda, \Lambda, \mathcal{D})$ .*

*Proof.* See proof of [6, Lemma 2.20] and Lemma 4.1.  $\blacksquare$

**Lemma 2.5.** Let  $\bar{\mathbf{A}} = (\bar{a}^{ij})$  be a constant symmetric matrix satisfying (1.10). For  $\mathbf{g} \in L^2(\mathcal{D})$  let  $u \in L^2(\mathcal{D})$  be a unique adjoint solution to

$$\sum_{i,j=1}^n D_{ij}(\bar{a}^{ij}u) = \operatorname{div}^2 \mathbf{g} \text{ in } \mathcal{D}; \quad u = \frac{\mathbf{g}^v \cdot \nu}{\bar{\mathbf{A}}^v \cdot \nu} \text{ on } \partial\mathcal{D}.$$

Then for any  $t > 0$ , we have

$$|\{x \in \mathcal{D} : |u(x)| > t\}| \leq \frac{C}{t} \int_{\mathcal{D}} |g|,$$

where  $C = C(n, \lambda, \Lambda, \mathcal{D})$ .

*Proof.* See proof of [6, Lemma 2.23] and Lemma 4.1.  $\blacksquare$

We finish this subsection by a Lipschitz estimate for the following equation, which will be used in the proof of Theorem 1.17:

$$D_{ij}(\bar{a}^{ij}u) = \operatorname{div}^2 \bar{\mathbf{g}} \text{ in } B^+(0, 2), \quad u = \frac{\bar{\mathbf{g}}^v \cdot \nu}{\bar{\mathbf{A}}^v \cdot \nu} \text{ on } T(0, 2), \quad (2.6)$$

where  $\bar{\mathbf{A}} = (\bar{a}^{ij})$  and  $\bar{\mathbf{g}} = (\bar{g}^{ij})$  are constant symmetric matrices.

**Lemma 2.7.** Let us denote  $B_r^+ := B^+(0, r)$ . Suppose that  $u \in L^2(B_2^+)$  satisfies (2.6). Then for any  $p \in (0, 1)$  and  $c \in \mathbb{R}$ , we have

$$\|Du\|_{L^\infty(B_1^+)} \leq C\|u - c\|_{L^p(B_2^+)}, \quad (2.8)$$

where  $C = C(n, \lambda, \Lambda, p)$ .

*Proof.* First we notice that  $u$  is smooth in  $B^+(0, 2) \cup T(0, 2)$  and satisfies

$$\bar{a}_{ij}D_{ij}u = 0 \text{ in } B^+(0, 2), \quad u = \text{constant on } T(0, 2).$$

Obviously,  $u - c$  enjoys the same properties for any  $c \in \mathbb{R}$ . Thus, without loss of generality, we may assume that  $c = 0$ . By a linear transformation, we may further assume that  $\bar{a}_{ij} = \delta_{ij}$ , i.e.  $\bar{\mathbf{A}} = \mathbf{I}$ . The problem is thus reduced to

$$\Delta u = 0 \text{ in } B^+(0, 2), \quad u = \text{constant on } T(0, 2). \quad (2.9)$$

By differentiating (2.9) in the tangential direction  $x_k$  for  $k = 1, 2, \dots, n-1$ , we see that  $v_k = D_k u$  satisfies

$$\Delta v_k = 0 \text{ in } B^+(0, 2), \quad v_k = 0 \text{ on } T(0, 2).$$

By classical estimates for harmonic functions, we thus have

$$\|D_k u\|_{L^\infty(B_1^+)} + \|DD_k u\|_{L^\infty(B_1^+)} \leq C\|D_k u\|_{L^2(B_{3/2}^+)}, \quad k = 1, 2, \dots, n-1. \quad (2.10)$$

Next, from the equation, we find that  $D_{nn}u = -\sum_{k=1}^{n-1} D_{kk}u = 0$  on  $T(0, 2)$ . Therefore, the normal derivative  $v_n = D_n u$  satisfies

$$\Delta v_n = 0 \text{ in } B^+(0, 2), \quad D_n v_n = 0 \text{ on } T(0, 2).$$

Again, by classical estimates for harmonic functions, we have

$$\|D_n u\|_{L^\infty(B_1^+)} + \|DD_n u\|_{L^\infty(B_1^+)} \leq C\|D_n u\|_{L^2(B_{3/2}^+)}. \quad (2.11)$$

Combining (2.10) and (2.11) yields

$$\|Du\|_{L^\infty(B_1^+)} + \|D^2 u\|_{L^\infty(B_1^+)} \leq C\|Du\|_{L^2(B_{3/2}^+)}. \quad (2.12)$$

Now, (2.8) follows from (2.12), the interpolation inequality:

$$\|Du\|_{L^2(B_{3/2}^+)} \leq \varepsilon \|D^2u\|_{L^\infty(B_{3/2}^+)} + C(\varepsilon, n, p) \|u\|_{L^p(B_{3/2}^+)},$$

and a standard iteration argument.  $\blacksquare$

**2.2. Proof of Theorem 1.8.** First, we develop an interior  $C^1$  estimate.

**Proposition 2.13.** *For any  $\Omega' \subset\subset \Omega$ , we have  $u \in C^1(\overline{\Omega}')$ .*

*Proof.* By  $W^{1,p}$  theory, we have  $u \in W^{1,p}(\Omega)$  for any  $1 < p < \infty$  and

$$\|u\|_{W^{1,p}(\Omega)} \leq C\|f\|_{L^q(\Omega)} + C\|g\|_{L^\infty(\Omega)} + C\|u\|_{L^1(\Omega)}.$$

Here and below,  $C$  denotes a constant depending only on  $n, \lambda, \Lambda, p, q, \Omega, \partial\Omega$ , and the coefficients of  $L$ ; see, for instance, [5, Lemma 4]. Therefore, by the Morrey-Sobolev embedding,  $u \in C^{0,\mu}(\Omega)$  for any  $0 < \mu < 1$  and

$$\|u\|_{C^{0,\mu}(\Omega)} \leq C\|f\|_{L^q(\Omega)} + C\|g\|_{L^\infty(\Omega)} + C\|u\|_{L^1(\Omega)}.$$

In particular, we have

$$\varrho_u(t) \leq C\left(\|f\|_{L^q(\Omega)} + \|g\|_{L^\infty(\Omega)} + \|u\|_{L^1(\Omega)}\right)t^\mu.$$

We rewrite the equation as

$$D_i(a^{ij}D_ju) = f - c^iD_iu - du + D_i(g^i - ub^i).$$

We set  $g' = g - ub$ . Then  $g'$  is of Dini mean oscillation by Lemma 2.1. Moreover, by (2.2), we have

$$\omega_{g'}(t) \leq \omega_g(t) + C\left(\|f\|_{L^q(\Omega)} + \|g\|_{L^\infty(\Omega)} + \|u\|_{L^1(\Omega)}\right)\left(\omega_b(t) + \|b\|_{L^\infty(\Omega)}t^\mu\right).$$

We also set  $g'' = \nabla v$ , where  $v$  solves

$$\Delta v = f - c^iD_iu - du \quad \text{in } \Omega; \quad v = 0 \quad \text{on } \partial\Omega.$$

Note that  $f - c^iD_iu - du \in L^p(\Omega)$  for  $n < p < q$  and

$$\|f - c^iD_iu - du\|_{L^p(\Omega)} \leq C\left(\|f\|_{L^q(\Omega)} + \|g\|_{L^\infty(\Omega)} + \|u\|_{L^1(\Omega)}\right)\left(1 + \|c\|_{L^q(\Omega)} + \|d\|_{L^q(\Omega)}\right).$$

By the Calderón-Zygmund theory and the Sobolev-Morrey inequality, we find  $g'' \in C^{0,\delta}(\Omega)$  with  $\delta = 1 - \frac{n}{p}$  and

$$\omega_{g''}(t) \leq C\left(\|f\|_{L^q(\Omega)} + \|g\|_{L^\infty(\Omega)} + \|u\|_{L^1(\Omega)}\right)\left(1 + \|c\|_{L^q(\Omega)} + \|d\|_{L^q(\Omega)}\right)t^\delta.$$

Therefore, we find  $u$  is a weak solution of

$$\operatorname{div}(\mathbf{A}\nabla u) = \operatorname{div}(g' + g'') \quad \text{in } \Omega,$$

where  $g' + g''$  is of Dini mean oscillation. We have shown that  $\omega_{g'} + \omega_{g''}$  is a Dini function that is completely determined by the given data (namely  $n, \lambda, \Lambda, \Omega, \omega_A, p, q, \|f\|_{L^q(\Omega)}, \|c\|_{L^q(\Omega)}, \|d\|_{L^q(\Omega)}, \omega_b, \|b\|_{L^\infty(\Omega)}, \omega_g$ , and  $\|g\|_{L^\infty(\Omega)}$ ) and  $\|u\|_{L^1(\Omega)}$ . By [6, Theorem 1.5], we thus find that  $u \in C^1(\overline{\Omega}')$  and  $\|u\|_{C^1(\overline{\Omega}')}$  is bounded by a constant  $C$  depending only on the above mentioned given data,  $\|u\|_{L^1(\Omega)}$ , and  $\Omega'$ .  $\blacksquare$



Next, we turn to  $C^1$  estimate near the boundary. We shall write  $B_r^+ = B^+(0, r)$ . Let  $\mathbf{g}'$  and  $\mathbf{g}''$  be as given in the proof of Proposition 2.13. Under a mapping of flattening boundary

$$y = \Phi(x) = (\Phi^1(x), \dots, \Phi^n(x)), \quad (\det D\Phi = 1)$$

let  $\tilde{u}(y) = u(x)$ , which satisfies

$$D_i(\tilde{a}^{ij}D_j\tilde{u}) = \operatorname{div}(\tilde{\mathbf{g}}' + \tilde{\mathbf{g}}'')$$

and

$$\tilde{a}^{ij}(y) = D_i\Phi^j D_k\Phi^l a^{kl}(x), \quad \tilde{\mathbf{g}}'(y) = D\Phi^\top \mathbf{g}'(x), \quad \tilde{\mathbf{g}}''(y) = D\Phi^\top \mathbf{g}''(x).$$

By Lemma 2.1, we see that the coefficients  $\tilde{a}^{ij}$  as well as the data  $\tilde{\mathbf{g}}'$  and  $\tilde{\mathbf{g}}''$  are still of Dini mean oscillation. Therefore, we are reduced to prove the following.

**Proposition 2.14.** *If  $u \in W^{1,2}(B_4^+)$  is a weak solution of*

$$D_i(a^{ij}D_j u) = \operatorname{div} \mathbf{g} \text{ in } B_4^+$$

*satisfying  $u = 0$  on  $T(0, 4)$ , then  $u \in C^1(\bar{B}_1^+)$ .*

The rest of this subsection is devoted to the proof of Proposition 2.14. The proof of Proposition 2.14 is in the spirit of Campanato's method [3] as presented in a modern textbook [12]. We shall derive an a priori estimate of the modulus of continuity of  $Du$  by assuming that  $u$  is in  $C^1(\bar{B}_3^+)$ . The general case follows from a standard approximation argument.

Fix any  $p \in (0, 1)$ . For  $x \in B_4^+$  and  $r > 0$ , we define

$$\phi(x, r) := \inf_{q \in \mathbb{R}^n} \left( \int_{B(x, r) \cap B_4^+} |Du - q|^p \right)^{\frac{1}{p}} \quad (2.15)$$

and choose a vector  $q_{x,r} \in \mathbb{R}^n$  satisfying

$$\phi(x, r) = \left( \int_{B(x, r) \cap B_4^+} |Du - q_{x,r}|^p \right)^{\frac{1}{p}}. \quad (2.16)$$

Also, for  $\bar{x} \in T(0, 4)$  and  $r > 0$ , we introduce an auxiliary quantity

$$\varphi(\bar{x}, r) := \inf_{q \in \mathbb{R}^n} \left( \int_{B^+(\bar{x}, r) \cap B_4^+} |Du - q e_n|^p \right)^{\frac{1}{p}} \quad (e_n = (0, \dots, 0, 1) \in \mathbb{R}^n) \quad (2.17)$$

and fix a number  $\bar{q}_{\bar{x}, r} \in \mathbb{R}$  satisfying

$$\varphi(\bar{x}, r) = \left( \int_{B^+(\bar{x}, r) \cap B_4^+} |Du - \bar{q}_{\bar{x}, r} e_n|^p \right)^{\frac{1}{p}}. \quad (2.18)$$

We present a series of lemmas (and their proofs) that will provide key estimates for the proof of Proposition 2.14.

**Lemma 2.19.** *For any  $\bar{x} \in T(0, 3)$  and  $0 < \rho \leq r \leq \frac{1}{2}$ , we have*

$$\varphi(\bar{x}, \rho) \leq 2 \left( \frac{\rho}{r} \right)^\beta r^{-n} \|Du\|_{L^1(B^+(\bar{x}, r))} + C \|Du\|_{L^\infty(B^+(\bar{x}, 2r))} \tilde{\omega}_A(2\rho) + C \tilde{\omega}_g(2\rho), \quad (2.20)$$

where  $\beta \in (0, 1)$  and  $C > 0$  are constants depending only on  $n, \lambda, \Lambda$ , and  $p$ , and  $\tilde{\omega}_\bullet(t)$  is a Dini function derived from  $\omega_\bullet(t)$ .

*Proof.* Note that we have  $B^+(\bar{x}, 2r) \subset B_4^+$  and

$$\varphi(\bar{x}, r) \leq \left( \int_{B^+(\bar{x}, r)} |Du|^p \right)^{\frac{1}{p}} \lesssim r^{-n} \|Du\|_{L^1(B^+(\bar{x}, r))}. \quad (2.21)$$

We decompose  $u = v + w$ , where  $w \in W_0^{1,2}(\mathcal{D}(\bar{x}, 2r))$  is the solution of the problem

$$\operatorname{div}(\bar{\mathbf{A}} \nabla w) = -\operatorname{div}((\mathbf{A} - \bar{\mathbf{A}}) \nabla u) + \operatorname{div}(\mathbf{g} - \bar{\mathbf{g}}) \text{ in } \mathcal{D}(\bar{x}, 2r); \quad w = 0 \text{ on } \partial \mathcal{D}(\bar{x}, 2r).$$

Here and below, we use the simplified notation

$$\bar{\mathbf{A}} = \bar{\mathbf{A}}_{B^+(\bar{x}, 2r)}, \quad \bar{\mathbf{g}} = \bar{\mathbf{g}}_{B^+(\bar{x}, 2r)}.$$

By Lemma 2.3 with scaling, for any  $t > 0$ , we have

$$|\{x \in B^+(\bar{x}, r) : |Dw(x)| > t\}| \lesssim \frac{1}{t} \left( \|Du\|_{L^\infty(B^+(\bar{x}, 2r))} \int_{B^+(\bar{x}, 2r)} |\mathbf{A} - \bar{\mathbf{A}}| + \int_{B^+(\bar{x}, 2r)} |\mathbf{g} - \bar{\mathbf{g}}| \right),$$

where we used  $B^+(\bar{x}, r) \subset \mathcal{D}(\bar{x}, 2r) \subset B^+(\bar{x}, 2r)$ . Then, we have (cf. [6, (2.11)])

$$\left( \int_{B^+(\bar{x}, r)} |Dw|^p \right)^{\frac{1}{p}} \lesssim \omega_{\mathbf{A}}(2r) \|Du\|_{L^\infty(B^+(\bar{x}, 2r))} + \omega_{\mathbf{g}}(2r). \quad (2.22)$$

On the other hand,  $v = u - w$  satisfies

$$\operatorname{div}(\bar{\mathbf{A}} \nabla v) = 0 \text{ in } B^+(\bar{x}, r); \quad v = 0 \text{ on } T(\bar{x}, r). \quad (2.23)$$

Note that the same is satisfied by  $D_j v$  for  $j = 1, \dots, n-1$ . By standard boundary estimates for elliptic equations (or systems) with constant coefficients, we have

$$\|DD_j v\|_{L^\infty(B^+(\bar{x}, \frac{1}{2}r))} \lesssim r^{-1} \left( \int_{B^+(\bar{x}, r)} |D_j v|^p \right)^{\frac{1}{p}} \lesssim r^{-1} \left( \int_{B^+(\bar{x}, r)} |D_{x'} v|^p \right)^{\frac{1}{p}},$$

where  $|D_{x'} v|^2 := \sum_{j=1}^{n-1} (D_j v)^2$ . Since

$$D_{nm} v = \frac{1}{\bar{a}^{nm}} \sum_{i=1}^n \sum_{j=1}^{n-1} \bar{a}^{ij} D_{ij} v,$$

we obtain

$$\|D^2 v\|_{L^\infty(B^+(\bar{x}, \frac{1}{2}r))} \lesssim r^{-1} \left( \int_{B^+(\bar{x}, r)} |D_{x'} v|^p \right)^{\frac{1}{p}}. \quad (2.24)$$

Therefore, we have

$$\|D^2 v\|_{L^\infty(B^+(\bar{x}, \frac{1}{2}r))} \lesssim r^{-1} \left( \int_{B^+(\bar{x}, r)} |Dv - qe_n|^p \right)^{\frac{1}{p}}, \quad \forall q \in \mathbb{R}. \quad (2.25)$$

Let  $0 < \kappa < \frac{1}{2}$  to be a number to be fixed later. Note that we have

$$\left( \int_{B^+(\bar{x}, \kappa r)} |D_n v - \overline{D_n v}_{B^+(\bar{x}, \kappa r)}|^p \right)^{\frac{1}{p}} \leq 2\kappa r \|D^2 v\|_{L^\infty(B(\bar{x}, \frac{1}{2}r))},$$

while, for  $j = 1, \dots, n-1$ , we have

$$\left( \int_{B^+(\bar{x}, \kappa r)} |D_j v|^p \right)^{\frac{1}{p}} = \left( \int_{B^+(\bar{x}, \kappa r)} |D_j v - D_j v(\bar{x})|^p \right)^{\frac{1}{p}} \leq 2\kappa r \|D^2 v\|_{L^\infty(B(\bar{x}, \frac{1}{2}r))}.$$

Hence, by (2.25) we obtain

$$\left( \int_{B^+(\bar{x}, \kappa r)} |Dv - \overline{D_n v}_{B^+(\bar{x}, \kappa r)} e_n|^p \right)^{\frac{1}{p}} \leq C_0 \kappa \left( \int_{B^+(\bar{x}, r)} |Dv - q e_n|^p \right)^{\frac{1}{p}}, \quad \forall q \in \mathbb{R}, \quad (2.26)$$

where  $C_0$  is an absolute constant determined only by  $n, \lambda, \Lambda$ , and  $p$ . By using the decomposition  $u = v + w$ , we obtain from (2.26) that

$$\begin{aligned} & \left( \int_{B^+(\bar{x}, \kappa r)} |Du - \overline{D_n u}_{B^+(\bar{x}, \kappa r)} e_n|^p \right)^{\frac{1}{p}} \\ & \leq 2^{\frac{1-p}{p}} \left( \int_{B^+(\bar{x}, \kappa r)} |Dv - \overline{D_n v}_{B^+(\bar{x}, \kappa r)} e_n|^p \right)^{\frac{1}{p}} + C \left( \int_{B^+(\bar{x}, \kappa r)} |Dw|^p \right)^{\frac{1}{p}} \\ & \leq 4^{\frac{1-p}{p}} C_0 \kappa \left( \int_{B^+(\bar{x}, r)} |Du - q e_n|^p \right)^{\frac{1}{p}} + C(\kappa^{-\frac{n}{p}} + 1) \left( \int_{B^+(\bar{x}, r)} |Dw|^p \right)^{\frac{1}{p}}. \end{aligned}$$

Since  $q \in \mathbb{R}$  is arbitrary, by using (2.22), we thus obtain

$$\varphi(\bar{x}, \kappa r) \leq 4^{\frac{1-p}{p}} C_0 \kappa \varphi(\bar{x}, r) + C(\kappa^{-\frac{n}{p}} + 1) (\omega_A(2r) \|Du\|_{L^\infty(B^+(\bar{x}, 2r))} + \omega_g(2r)).$$

Now we choose  $\kappa$  sufficiently small<sup>2</sup> so that  $4^{\frac{1-p}{p}} C_0 \kappa \leq \frac{1}{2}$ . Then, we obtain

$$\varphi(\bar{x}, \kappa r) \leq \frac{1}{2} \varphi(\bar{x}, r) + C (\omega_A(2r) \|Du\|_{L^\infty(B^+(\bar{x}, 2r))} + \omega_g(2r)).$$

By iterating, for  $j = 1, 2, \dots$ , we get

$$\varphi(\bar{x}, \kappa^j r) \leq 2^{-j} \varphi(\bar{x}, r) + C \|Du\|_{L^\infty(B^+(\bar{x}, 2r))} \sum_{i=1}^j 2^{1-i} \omega_A(2\kappa^i r) + C \sum_{i=1}^j 2^{1-i} \omega_g(2\kappa^i r).$$

Therefore, we have

$$\varphi(\bar{x}, \kappa^j r) \leq 2^{-j} \varphi(\bar{x}, r) + C \|Du\|_{L^\infty(B^+(\bar{x}, 2r))} \tilde{\omega}_A(2\kappa^j r) + C \tilde{\omega}_g(2\kappa^j r), \quad (2.27)$$

where we set

$$\tilde{\omega}_\bullet(t) = \sum_{i=1}^{\infty} \frac{1}{2^i} (\omega_\bullet(\kappa^{-i} t) [\kappa^{-i} t \leq 1] + \omega_\bullet(1) [\kappa^{-i} t > 1]). \quad (2.28)$$

Here, we used Iverson bracket notation; i.e.,  $[P] = 1$  if  $P$  is true and  $[P] = 0$  otherwise. We recall that  $\tilde{\omega}_\bullet(t)$  is a Dini function; see [5, Lemma 1].

Now, for any  $\rho$  satisfying  $0 < \rho \leq r$ , we take  $j$  to be the integer satisfying  $\kappa^{j+1} < \rho/r \leq \kappa^j$  and set  $\beta = \frac{\ln \frac{1}{\kappa}}{\ln \kappa}$ . It should be clear that  $0 < \beta < 1$ . Then, by (2.27)

$$\varphi(\bar{x}, \rho) \leq 2 \left( \frac{\rho}{r} \right)^\beta \varphi(\bar{x}, \kappa^{-j} \rho) + C \|Du\|_{L^\infty(B^+(\bar{x}, 2r))} \tilde{\omega}_A(2\rho) + C \tilde{\omega}_g(2\rho). \quad (2.29)$$

Therefore, we get (2.20) from (2.29) and (2.21).  $\blacksquare$

**Lemma 2.30.** *For any  $x \in B_3^+$  and  $0 < \rho \leq r \leq \frac{1}{4}$ , we have*

$$\phi(x, \rho) \leq C \left( \frac{\rho}{r} \right)^\beta r^{-n} \|Du\|_{L^1(B(x, 3r) \cap B_4^+)} + C \|Du\|_{L^\infty(B(x, 5r) \cap B_4^+)} \hat{\omega}_A(\rho) + C \hat{\omega}_g(\rho), \quad (2.31)$$

where  $\beta \in (0, 1)$  and  $C > 0$  are constants depending only on  $n, \lambda, \Lambda$ , and  $p$ , and  $\hat{\omega}_\bullet(t)$  is a Dini function derived from  $\omega_\bullet(t)$ .

<sup>2</sup>We may assume that  $\kappa$  agrees with the one introduced in the proof of [6, Theorem 1.5].

*Proof.* In this proof we shall denote

$$\bar{x} = (x^1, \dots, x^{n-1}, 0).$$

There are three possibilities.

- i.  $\rho \leq r \leq x^n$ : We utilize an interior  $C^1$  estimate developed in [6] as follows. Since  $B(x, r) \subset B_4^+$ , we observe that  $\phi(x, \rho)$  is identical to that introduced in the proof of [6, Theorem 1.5] and satisfies (c.f. [6, (2.14)])

$$\phi(x, \kappa^j r) \leq 2^{-j} \phi(x, r) + C \|Du\|_{L^\infty(B(x, r))} \tilde{\omega}_A(\kappa^j r) + C \tilde{\omega}_g(\kappa^j r), \quad (2.32)$$

where  $\kappa \in (0, \frac{1}{2})$  and  $\tilde{\omega}_\bullet$  are as in (2.27) and (2.28), respectively. Then we get an inequality similar to (2.29), namely, for  $0 < \rho \leq r \leq \frac{1}{4}$

$$\phi(x, \rho) \leq 2 \left( \frac{\rho}{r} \right)^\beta \phi(x, \kappa^{-j} \rho) + C \|Du\|_{L^\infty(B(x, r))} \tilde{\omega}_A(\rho) + C \tilde{\omega}_g(\rho), \quad (2.33)$$

where  $j$  is the integer satisfying  $\kappa^{j+1} < \rho/r \leq \kappa^j$ . By (2.15), we have

$$\phi(x, \kappa^{-j} \rho) \leq \left( \int_{B(x, \kappa^{-j} \rho)} |Du|^p \right)^{\frac{1}{p}} \leq \kappa^{-n} \int_{B(x, r)} |Du|,$$

and thus, we obtain

$$\phi(x, \rho) \leq C \left( \frac{\rho}{r} \right)^\beta r^{-n} \|Du\|_{L^1(B(x, r))} + C \|Du\|_{L^\infty(B(x, r))} \tilde{\omega}_A(\rho) + C \tilde{\omega}_g(\rho). \quad (2.34)$$

- ii.  $x^n \leq \rho \leq r$ : Since  $B(x, \rho) \cap B_4^+ \subset B^+(\bar{x}, 2\rho) \subset B_4^+$ , we have

$$\begin{aligned} \phi(x, \rho) &= \left( \int_{B(x, \rho) \cap B^+(0, 4)} |Du - q_{x, \rho}|^p \right)^{\frac{1}{p}} \leq \left( \int_{B(x, \rho) \cap B^+(0, 4)} |Du - \bar{q}_{\bar{x}, 2\rho} e_n|^p \right)^{\frac{1}{p}} \\ &\leq 2^{\frac{n}{p}} \left( \int_{B^+(\bar{x}, 2\rho)} |Du - \bar{q}_{\bar{x}, 2\rho} e_n|^p \right)^{\frac{1}{p}} = 2^{\frac{n}{p}} \phi(\bar{x}, 2\rho). \end{aligned} \quad (2.35)$$

Therefore, by Lemma 2.19, and using  $|x - \bar{x}| = x^n \leq r$ , we obtain

$$\begin{aligned} \phi(x, \rho) &\leq C \left( \frac{2\rho}{2r} \right)^\beta r^{-n} \|Du\|_{L^1(B^+(\bar{x}, 2r))} + C \|Du\|_{L^\infty(B^+(\bar{x}, 4r))} \tilde{\omega}_A(4\rho) + C \tilde{\omega}_g(4\rho) \\ &\leq C \left( \frac{\rho}{r} \right)^\beta r^{-n} \|Du\|_{L^1(B(x, 3r) \cap B_4^+)} + C \|Du\|_{L^\infty(B(x, 5r) \cap B_4^+)} \tilde{\omega}_A(4\rho) + C \tilde{\omega}_g(4\rho). \end{aligned} \quad (2.36)$$

- iii.  $\rho \leq x^n \leq r$ : Take  $R = x^n$  and let  $j$  be the integer satisfying  $\kappa^{j+1} < \rho/R \leq \kappa^j$ . Since  $B(x, \kappa^{-j} \rho) \subset B(x, R) \subset B_4^+$  and  $B(x, R) \subset B^+(\bar{x}, 2R)$ , we have

$$\begin{aligned} \phi(x, \kappa^{-j} \rho) &= \left( \int_{B(x, \kappa^{-j} \rho)} |Du - q_{x, \kappa^{-j} \rho}|^p \right)^{\frac{1}{p}} \leq \left( \int_{B(x, \kappa^{-j} \rho)} |Du - \bar{q}_{\bar{x}, 2R} e_n|^p \right)^{\frac{1}{p}} \\ &\leq 2^{\frac{n-1}{p}} \kappa^{-\frac{n}{p}} \left( \int_{B^+(\bar{x}, 2R)} |Du - \bar{q}_{\bar{x}, 2R} e_n|^p \right)^{\frac{1}{p}} = 2^{\frac{n-1}{p}} \kappa^{-\frac{n}{p}} \phi(\bar{x}, 2R). \end{aligned} \quad (2.37)$$

Therefore, by (2.33) and Lemma 2.19, we get

$$\begin{aligned}\phi(x, \rho) &\leq C \left( \frac{\rho}{R} \right)^\beta \varphi(\bar{x}, 2R) + C \|Du\|_{L^\infty(B(x, R))} \tilde{\omega}_A(\rho) + C \tilde{\omega}_g(\rho) \\ &\leq C \left( \frac{\rho}{R} \right)^\beta \left\{ \left( \frac{2R}{2r} \right)^\beta r^{-n} \|Du\|_{L^1(B^+(\bar{x}, 2r))} + \|Du\|_{L^\infty(B^+(\bar{x}, 4r))} \tilde{\omega}_A(4R) + \tilde{\omega}_g(4R) \right\} \\ &\quad + C \|Du\|_{L^\infty(B^+(\bar{x}, r))} \tilde{\omega}_A(\rho) + C \tilde{\omega}_g(\rho).\end{aligned}$$

Therefore, by setting

$$\omega_\bullet^\sharp(t) := \sup_{s \in [t, 1]} \left( \frac{t}{s} \right)^\beta \tilde{\omega}_\bullet(s) \quad (0 < t \leq 1)$$

and using  $|x - \bar{x}| = x^n \leq r$ , we obtain

$$\begin{aligned}\phi(x, \rho) &\leq C \left( \frac{\rho}{r} \right)^\beta r^{-n} \|Du\|_{L^1(B(x, 3r) \cap B_4^+)} + C \|Du\|_{L^\infty(B(x, 5r) \cap B_4^+)} \omega_A^\sharp(4\rho) + C \omega_g^\sharp(4\rho) \\ &\quad + C \|Du\|_{L^\infty(B(x, 2r) \cap B_4^+)} \tilde{\omega}_A(\rho) + C \tilde{\omega}_g(\rho).\end{aligned}\tag{2.38}$$

We have covered all three possible cases and obtained bounds for  $\phi(x, \rho)$ , namely, (2.34), (2.38), and (2.36). Therefore, if we set  $\hat{\omega}_\bullet(t)$  as

$$\hat{\omega}_\bullet(t) := \tilde{\omega}_\bullet(t) + \tilde{\omega}_\bullet(4t) + \omega_\bullet^\sharp(4t),$$

then (2.31) follows. To complete the proof, we only need to show that  $\omega_\bullet^\sharp(t)$  is a Dini function. By [5, Lemma 1], it is enough to show

$$\omega_\bullet^\sharp(t) \lesssim \sum_{j=0}^{\infty} \frac{1}{2^{j\beta}} \left( \tilde{\omega}_\bullet(2^j t) [2^j t \leq 1] + \tilde{\omega}_\bullet(1) [2^j t > 1] \right).\tag{2.39}$$

To see this, we first recall  $\tilde{\omega}_\bullet$  satisfies the property (1.4). See Remark 2.40 below. Since for any  $s \in [t, 1]$ , there is an integer  $j$  be an integer such that  $2^{j-1}t \leq s < 2^j t$  and (2.39) follows from the definition of  $\omega_\bullet^\sharp(t)$ .  $\blacksquare$

*Remark 2.40.* It can be easily seen that  $\tilde{\omega}_\bullet$  satisfies the condition (1.4); see [6]. From the construction of  $\hat{\omega}_\bullet$  in the above proof, it is routine to verify that  $\hat{\omega}_\bullet$  satisfies the property (1.4) as well.

**Lemma 2.41.** *We have*

$$\|Du\|_{L^\infty(B_2^+)} \leq C \|Du\|_{L^1(B_4^+)} + C \int_0^1 \frac{\hat{\omega}_g(t)}{t} dt,\tag{2.42}$$

where  $C > 0$  is a constant depending only on  $n, \lambda, \Lambda, p$ , and  $\omega_A$ .

*Proof.* For  $x \in B_3^+$  and  $0 < r \leq \frac{1}{4}$ , let  $\{q_{x, 2^{-k}r}\}_{k=0}^\infty$  be a sequence of vectors in  $\mathbb{R}^n$  as given in (2.16). Since we have

$$|q_{x,r} - q_{x, \frac{1}{2}r}|^p \leq |Du(y) - q_{x,r}|^p + |Du(y) - q_{x, \frac{1}{2}r}|^p,$$

by taking average over  $y \in B(x, \frac{1}{2}r) \cap B_4^+$  and then taking  $p$ th root, we obtain

$$|q_{x,r} - q_{x, \frac{1}{2}r}| \lesssim \phi(x, r) + \phi(x, \frac{1}{2}r).\tag{2.43}$$

Then, by iterating, we get

$$|q_{x, 2^{-k}r} - q_{x,r}| \lesssim \sum_{j=0}^k \phi(x, 2^{-j}r).\tag{2.44}$$

Note that (2.31) implies

$$\lim_{k \rightarrow \infty} \phi(x, 2^{-k}r) = 0,$$

and thus, by the assumption that  $u \in C^1(\bar{B}_3^+)$ , we find

$$\lim_{k \rightarrow \infty} q_{x, 2^{-k}r} = Du(x).$$

Therefore, by taking  $k \rightarrow \infty$  in (2.44), using (2.31) and Remark 2.40, we get

$$|Du(x) - q_{x,r}| \lesssim r^{-n} \|Du\|_{L^1(B(x,3r) \cap B_4^+)} + \|Du\|_{L^\infty(B(x,5r) \cap B_4^+)} \int_0^r \frac{\hat{\omega}_A(t)}{t} dt + \int_0^r \frac{\hat{\omega}_g(t)}{t} dt.$$

By averaging the obvious inequality

$$|q_{x,r}|^p \leq |Du(y) - q_{x,r}|^p + |Du(y)|^p$$

over  $y \in B(x, r) \cap B_4^+$  and taking  $p$ th root, we get

$$|q_{x,r}| \lesssim \phi(x, r) + \left( \int_{B(x,r) \cap B_4^+} |Du|^p \right)^{\frac{1}{p}}.$$

Combining these together and using

$$\phi(x, r) \lesssim r^{-n} \|Du\|_{L^1(B(x,r) \cap B_4^+)},$$

we obtain

$$|Du(x)| \lesssim r^{-n} \|Du\|_{L^1(B(x,3r) \cap B_4^+)} + \|Du\|_{L^\infty(B(x,5r) \cap B_4^+)} \int_0^r \frac{\hat{\omega}_A(t)}{t} dt + \int_0^r \frac{\hat{\omega}_g(t)}{t} dt.$$

Now, taking supremum for  $x \in B(x_0, r) \cap B_4^+$ , where  $x_0 \in B_3^+$  and  $r \leq \frac{1}{4}$ , we have

$$\begin{aligned} \|Du\|_{L^\infty(B(x_0,r) \cap B_4^+)} &\leq Cr^{-n} \|Du\|_{L^1(B(x_0,4r) \cap B_4^+)} \\ &\quad + C \|Du\|_{L^\infty(B(x_0,6r) \cap B_4^+)} \int_0^r \frac{\hat{\omega}_A(t)}{t} dt + C \int_0^r \frac{\hat{\omega}_g(t)}{t} dt. \end{aligned}$$

We fix  $r_0 < \frac{1}{4}$  such that for any  $0 < r \leq r_0$ ,

$$C \int_0^r \frac{\hat{\omega}_A(t)}{t} dt \leq \frac{1}{3^n}.$$

Then, we have for any  $x_0 \in B_3^+$  and  $0 < r \leq r_0$  that

$$\|Du\|_{L^\infty(B(x_0,r) \cap B_4^+)} \leq 3^{-n} \|Du\|_{L^\infty(B(x_0,6r) \cap B_4^+)} + Cr^{-n} \|Du\|_{L^1(B(x_0,4r) \cap B_4^+)} + C \int_0^r \frac{\hat{\omega}_g(t)}{t} dt.$$

For  $k = 1, 2, \dots$ , denote  $r_k = 3 - 2^{1-k}$ . Note that  $r_{k+1} - r_k = 2^{-k}$  for  $k \geq 1$  and  $r_1 = 2$ . For  $x_0 \in B_{r_k}^+$  and  $r \leq 2^{-k-3}$ , we have  $B(x_0, 6r) \cap B_4^+ \subset B_{r_{k+1}}^+$ . We take  $k_0$  sufficiently large such that  $2^{-k_0-3} \leq r_0$ . It then follows that for any  $k \geq k_0$ ,

$$\|Du\|_{L^\infty(B_{r_k}^+)} \leq 3^{-n} \|Du\|_{L^\infty(B_{r_{k+1}}^+)} + C 2^{kn} \|Du\|_{L^1(B_4^+)} + C \int_0^1 \frac{\hat{\omega}_g(t)}{t} dt.$$

By multiplying the above by  $3^{-kn}$  and then summing over  $k \geq k_0$ , we reach

$$\sum_{k=k_0}^{\infty} 3^{-kn} \|Du\|_{L^\infty(B_{r_k}^+)} \leq \sum_{k=k_0}^{\infty} 3^{-(k+1)n} \|Du\|_{L^\infty(B_{r_{k+1}}^+)} + C \|Du\|_{L^1(B_4^+)} + C \int_0^1 \frac{\hat{\omega}_g(t)}{t} dt.$$

Since we assume that  $u \in C^1(\bar{B}_3^+)$ , the summations on both sides are convergent and we obtain (2.42).  $\blacksquare$

With  $\tilde{\omega}_\bullet(t)$  and  $\hat{\omega}_\bullet(t)$  be as in Lemmas 2.19 and 2.30, we define

$$\omega_\bullet^*(t) := \hat{\omega}_\bullet(t) + \int_0^t \frac{\tilde{\omega}_\bullet(s)}{s} ds + \tilde{\omega}_\bullet(4t) + \int_0^t \frac{\tilde{\omega}_\bullet(4s)}{s} ds. \quad (2.45)$$

**Lemma 2.46.** *For any  $x \in B_3^+$  and  $0 < r \leq \frac{1}{5}$ , we have*

$$|Du(x) - q_{x,r}| \leq Cr^\beta \|Du\|_{L^1(B(x, \frac{3}{5}) \cap B_4^+)} + C\|Du\|_{L^\infty(B(x,1) \cap B_4^+)} \omega_\mathbf{A}^*(r) + C\omega_g^*(r), \quad (2.47)$$

where  $\beta \in (0, 1)$  and  $C > 0$  are constants depending only on  $n, \lambda, \Lambda$ , and  $p$ , and  $\omega^*(t)$  is defined as in (2.45).

*Proof.* As in the proof of Lemma 2.30, we denote  $\bar{x} = (x^1, \dots, x^{n-1}, 0)$  and  $\kappa < \frac{1}{2}$  be the same constant as (2.27) and (2.32). Let  $\{q_{x, \kappa^i r}\}_{i=0}^\infty \in \mathbb{R}^n$  and  $\{\bar{q}_{\bar{x}, 2\kappa^i r}\}_{i=0}^\infty \in \mathbb{R}$  be sequences that are chosen accordingly as in (2.16) and (2.18). By using

$$\lim_{i \rightarrow \infty} q_{x, \kappa^i r} = Du(x)$$

and a computation similar to (2.43), we get

$$|Du(x) - q_{x,r}| \leq \sum_{i=0}^\infty |q_{x, \kappa^i r} - q_{x, \kappa^{i+1} r}| \lesssim \sum_{i=0}^\infty \phi(x, \kappa^i r).$$

Therefore, it suffices to bound  $\sum_{i=0}^\infty \phi(x, \kappa^i r)$  by the right-hand side of (2.47).

In the case when  $r \leq x^n$ , by (2.32) and (1.4), we have

$$\sum_{i=0}^\infty \phi(x, \kappa^i r) \lesssim \phi(x, r) + \|Du\|_{L^\infty(B(x, \frac{1}{5}) \cap B_4^+)} \int_0^r \frac{\tilde{\omega}_\mathbf{A}(t)}{t} dt + \int_0^r \frac{\tilde{\omega}_g(t)}{t} dt.$$

By Lemma 2.30, we get

$$\phi(x, r) \lesssim r^\beta \|Du\|_{L^1(B(x, \frac{3}{5}) \cap B_4^+)} + \|Du\|_{L^\infty(B(x,1) \cap B_4^+)} \hat{\omega}_\mathbf{A}(r) + \hat{\omega}_g(r),$$

By combining the above two inequalities, we obtain (2.47).

In the case when  $r > x^n$ , let  $i_0$  be the integer such that  $\kappa^{i_0+1} r \leq x^n < \kappa^{i_0} r$ . By (2.32) and [6, Lemma 2.7], we have

$$\sum_{i=i_0+1}^\infty \phi(x, \kappa^i r) \lesssim \phi(x, \kappa^{i_0+1} r) + \|Du\|_{L^\infty(B(x,r) \cap B_4^+)} \int_0^r \frac{\tilde{\omega}_\mathbf{A}(t)}{t} dt + \int_0^r \frac{\tilde{\omega}_g(t)}{t} dt.$$

By a computation similar to (2.37), we have

$$\phi(x, \kappa^{i_0+1} r) \leq 2^{\frac{n-1}{p}} \kappa^{-\frac{n}{p}} \varphi(\bar{x}, 2\kappa^{i_0} r)$$

and by (2.35), for  $0 \leq i \leq i_0$ , we have

$$\phi(x, \kappa^i r) \leq 2^{\frac{n}{p}} \varphi(\bar{x}, 2\kappa^i r).$$

Hence, we have

$$\sum_{i=0}^\infty \phi(x, \kappa^i r) \lesssim \sum_{i=0}^{i_0} \varphi(\bar{x}, 2\kappa^i r) + \|Du\|_{L^\infty(B(x,r) \cap B_4^+)} \int_0^r \frac{\tilde{\omega}_\mathbf{A}(t)}{t} dt + \int_0^r \frac{\tilde{\omega}_g(t)}{t} dt.$$

On the other hand, by (2.27) and assumption  $|x - \bar{x}| = x^n < r \leq \frac{1}{5}$ , we have

$$\sum_{i=0}^{\infty} \varphi(\bar{x}, 2\kappa^i r) \lesssim \varphi(\bar{x}, 2r) + \|Du\|_{L^\infty(B(x,1) \cap B_4^+)} \int_0^r \frac{\tilde{\omega}_A(4t)}{t} dt + \int_0^r \frac{\tilde{\omega}_g(4t)}{t} dt.$$

By Lemma 2.19 and using  $|x - \bar{x}| \leq \frac{1}{5}$ , we find

$$\varphi(\bar{x}, 2r) \lesssim r^\beta \|Du\|_{L^1(B(x, \frac{3}{5}) \cap B_4^+)} + \|Du\|_{L^\infty(B(x,1) \cap B_4^+)} \tilde{\omega}_A(4r) + \tilde{\omega}_g(4r).$$

Combining these together, we get (2.47) as well.  $\blacksquare$

Now, we are ready to show that  $u \in C^1(\bar{B}_1^+)$ . For  $x, y \in B_1^+$ , we have

$$|Du(x) - Du(y)| \leq |Du(x) - q_{x,r}| + |q_{x,r} - q_{y,r}| + |Du(y) - q_{y,r}|.$$

In the case when  $|x - y| < \frac{1}{5}$ , set  $r = |x - y|$  and apply Lemma 2.46 to get

$$|Du(x) - q_{x,r}| + |Du(y) - q_{y,r}| \lesssim r^\beta \|Du\|_{L^1(B_2^+)} + \|Du\|_{L^\infty(B_2^+)} \omega_A^*(r) + \omega_g^*(r).$$

Take the average over  $z \in B(x, r) \cap B(y, r) \cap B_4^+$  in the inequality

$$|q_{x,r} - q_{y,r}|^p \leq |Du(z) - q_{x,r}|^p + |Du(z) - q_{y,r}|^p$$

and take the  $p$ th root and apply Lemma 2.30 to get

$$|q_{x,r} - q_{y,r}| \lesssim \phi(x, r) + \phi(y, r) \lesssim r^\beta \|Du\|_{L^1(B_2^+)} + \|Du\|_{L^\infty(B_2^+)} \hat{\omega}_A(r) + \hat{\omega}_g(r).$$

Combining these together and using Lemma 2.41, we obtain (note  $\hat{\omega}_\bullet(t) \leq \omega_\bullet^*(t)$ )

$$\begin{aligned} |Du(x) - Du(y)| &\lesssim \|Du\|_{L^1(B_2^+)} |x - y|^\beta \\ &\quad + \left( \|Du\|_{L^1(B_4^+)} + \int_0^1 \frac{\hat{\omega}_g(t)}{t} dt \right) \omega_A^*(|x - y|) + \omega_g^*(|x - y|). \end{aligned} \quad (2.48)$$

In case when  $|x - y| \geq \frac{1}{5}$ , we use  $|Du(x) - Du(y)| \leq 2\|Du\|_{L^\infty(B_1^+)}$ , apply Lemma 2.41, and still obtain (2.48). This completes the proof of Proposition 2.14 and that of Theorem 1.8.  $\blacksquare$

**2.3. Proof of Theorem 1.12.** The idea of proof is essentially the same as that of Theorem 1.8. We first establish interior  $C^2$  estimates.

**Proposition 2.49.** *For any  $p \in (1, \infty)$ , we have  $u \in W^{2,p}(\Omega)$ . Moreover, for any  $\Omega' \subset\subset \Omega$ , we have  $u \in C^2(\bar{\Omega}')$ .*

*Proof.* By the  $W^{2,p}$  theory, we have  $u \in W^{2,p}(\Omega)$  for any  $1 < p < \infty$  and

$$\|u\|_{W^{2,p}(\Omega)} \leq C\|g\|_{L^\infty(\Omega)} + C\|u\|_{L^1(\Omega)},$$

where  $C$  is a constant depending only on  $n, \lambda, \Lambda, p, \Omega, \partial\Omega$ , and the coefficients of  $\mathcal{L}$ ; see, for instance, [13, Theorem 11.2.3]. Therefore, by the Morrey-Sobolev embedding,  $u \in C^{1,\mu}(\Omega)$  for any  $0 < \mu < 1$  and

$$\|u\|_{C^{1,\mu}(\Omega)} \leq C\|g\|_{L^\infty(\Omega)} + C\|u\|_{L^1(\Omega)}.$$

In particular, we have

$$\varrho_{Du}(t) + \varrho_u(t) \leq C\left(\|g\|_{L^\infty(\Omega)} + \|u\|_{L^1(\Omega)}\right)t^\mu.$$

We rewrite the equation as

$$a^{ij}D_{ij}u = g - b^iD_iu - cu =: g'.$$



Then  $g'$  is of Dini mean oscillation by Lemma 2.1. Moreover, by (2.2), we have

$$\omega_{g'}(t) \leq \omega_g(t) + C \left( \|g\|_{L^\infty(\Omega)} + \|u\|_{L^1(\Omega)} \right) \left\{ \omega_b(t) + \omega_c(t) + \left( \|b\|_{L^\infty(\Omega)} + \|c\|_{L^\infty(\Omega)} \right) t^\mu \right\}.$$

Therefore,  $\omega_{g'}$  is a Dini function that is completely determined by the given data (namely  $n, \lambda, \Lambda, \Omega, \omega_A, p, \omega_b, \|b\|_{L^\infty(\Omega)}, \omega_c, \omega_g$ , and  $\|g\|_{L^\infty(\Omega)}$ ) and  $\|u\|_{L^1(\Omega)}$ . By [6, Theorem 1.6], we thus find that  $u \in C^2(\overline{\Omega}')$  and  $\|u\|_{C^2(\overline{\Omega}')}$  is bounded by a constant  $C$  depending only on the above mentioned given data,  $\|u\|_{L^1(\Omega)}$ , and  $\Omega'$ . ■

Next, we turn to  $C^2$  estimate near the boundary. Let  $g'$  be as given in the proof of Proposition 2.49. Under a mapping of flattening boundary

$$y = \Phi(x) = (\Phi^1(x), \dots, \Phi^n(x)),$$

let  $\tilde{u}(y) = u(x)$ , which satisfies

$$\tilde{a}^{ij} D_{ij} \tilde{u} = \tilde{g}' - \tilde{b}^i D_i \tilde{u} =: \tilde{h},$$

where

$$\tilde{a}^{ij}(y) = D_l \Phi^i D_k \Phi^j a^{kl}(x), \quad \tilde{b}^i(y) = D_{kl} \Phi^i a^{kl}(x), \quad \tilde{g}'(y) = g'(x).$$

By Lemmas 2.1, we see that the coefficients  $\tilde{a}^{ij}$  and the data  $\tilde{h}$  are of Dini mean oscillation. As before, we are thus reduced to prove the following.

**Proposition 2.50.** *If  $u \in W^{2,2}(B_4^+)$  is a strong solution of*

$$a^{ij} D_{ij} u = g \quad \text{in } B_4^+$$

*satisfying  $u = 0$  on  $T(0, 4)$ , then  $u \in C^2(\overline{B}_1^+)$ .*

The rest of this subsection is devoted to the proof of Proposition 2.50. As in the proof of Proposition 2.14, we shall derive an a priori estimate of the modulus of continuity of  $D^2 u$  by assuming that  $u$  is in  $C^2(\overline{B}_3^+)$ .

Let  $\mathbb{S}(n)$  be the set of all symmetric  $n \times n$  matrices and let

$$\mathbb{S}_0(n) = \{\mathbf{q} = (q^{ij}) \in \mathbb{S}(n) : q^{ij} = 0 \text{ for } i, j = 1, 2, \dots, n-1\}.$$

Fix any  $p \in (0, 1)$ . Similar to (2.15) and (2.16), for  $x \in B_4^+$  and  $r > 0$ , we define

$$\phi(x, r) := \inf_{\mathbf{q} \in \mathbb{S}(n)} \left( \int_{B(x, r) \cap B_4^+} |D^2 u - \mathbf{q}|^p \right)^{\frac{1}{p}}$$

and fix a matrix  $\mathbf{q}_{x, r} \in \mathbb{S}(n)$  satisfying

$$\phi(x, r) = \left( \int_{B(x, r) \cap B_4^+} |D^2 u - \mathbf{q}_{x, r}|^p \right)^{\frac{1}{p}}.$$

Also, similar to (2.17), for  $\bar{x} \in T(0, 4)$  and  $r > 0$ , we introduce an auxiliary quantity

$$\varphi(\bar{x}, r) := \inf_{\mathbf{q} \in \mathbb{S}_0(n)} \left( \int_{B^+(\bar{x}, r)} |D^2 u - \mathbf{q}|^p \right)^{\frac{1}{p}}.$$

The following lemma is in parallel with Lemma 2.19.

**Lemma 2.51.** *For any  $\bar{x} \in T(0, 3)$  and  $0 < \rho \leq r \leq \frac{1}{2}$ , we have*

$$\varphi(\bar{x}, \rho) \leq 2 \left( \frac{\rho}{r} \right)^\beta r^{-n} \|D^2 u\|_{L^1(B^+(\bar{x}, r))} + C \|D^2 u\|_{L^\infty(B^+(\bar{x}, 2r))} \tilde{\omega}_A(2\rho) + C \tilde{\omega}_g(2\rho),$$

where  $\beta \in (0, 1)$  and  $C > 0$  are constants depending only on  $n, \lambda, \Lambda$ , and  $p$ , and  $\tilde{\omega}_\bullet(t)$  is a Dini function derived from  $\omega_\bullet(t)$ .

*Proof.* For  $\bar{x} \in T(0, 3)$  and  $0 < r \leq \frac{1}{2}$ , we decompose  $u = v + w$ , where  $w \in W^{2,2}(B(\bar{x}, r)) \cap W_0^{1,2}(B(\bar{x}, r))$  is a unique solution of the problem

$$\bar{a}^{ij} D_{ij} w = -\text{tr}((\mathbf{A} - \bar{\mathbf{A}}) D^2 u) + g - \bar{g} \quad \text{in } \mathcal{D}(\bar{x}, 2r); \quad w = 0 \quad \text{on } \partial \mathcal{D}(\bar{x}, 2r).$$

By Lemma 2.4 with scaling, we have for any  $\alpha > 0$ ,

$$|\{x \in B^+(\bar{x}, r) : |D^2 w(x)| > \alpha\}| \lesssim \frac{1}{\alpha} \left( \|D^2 u\|_{L^\infty(B^+(\bar{x}, 2r))} \int_{B^+(\bar{x}, 2r)} |\mathbf{A} - \bar{\mathbf{A}}| + \int_{B^+(\bar{x}, 2r)} |g - \bar{g}| \right).$$

Therefore, we have

$$\left( \int_{B^+(\bar{x}, r)} |D^2 w|^p \right)^{\frac{1}{p}} \lesssim \omega_A(2r) \|D^2 u\|_{L^\infty(B^+(\bar{x}, 2r))} + \omega_g(2r).$$

Since  $v = u - w$  satisfies

$$\bar{a}^{ij} D_{ij} v = D_i(\bar{a}^{ij} D_j v) = \bar{g} \quad \text{in } B^+(\bar{x}, r); \quad v = 0 \quad \text{on } T(\bar{x}, r),$$

we see that  $D_k v$  satisfies (2.23) for  $k = 1, \dots, n-1$ . Therefore, by (2.24), we have

$$\|D^2 D_k v\|_{L^\infty(B^+(\bar{x}, \frac{1}{2}r))} \lesssim r^{-1} \left( \int_{B^+(\bar{x}, r)} |D_{x'}^2 v|^p \right)^{\frac{1}{p}},$$

where  $|D_{x'}^2 v|^2 = \sum_{k,l=1}^{n-1} (D_{kl} v)^2$ . Since  $\bar{a}^{ij} D_{ij} (D_n v) = 0$ , we find

$$D_{nm} v = D_{nn} D_n v = \frac{1}{\bar{a}^{nn}} \sum_{i=1}^n \sum_{j=1}^{n-1} \bar{a}^{ij} D_{ijn} v,$$

and thus, we obtain

$$\|D^3 v\|_{L^\infty(B^+(\bar{x}, \frac{1}{2}r))} \lesssim r^{-1} \left( \int_{B^+(\bar{x}, r)} |D_{x'}^2 v|^p \right)^{\frac{1}{p}}.$$

Therefore, similar to (2.25), we have

$$\|D^3 v\|_{L^\infty(B^+(\bar{x}, \frac{1}{2}r))} \lesssim r^{-1} \left( \int_{B^+(\bar{x}, r)} |D^2 v - \mathbf{q}|^p \right)^{\frac{1}{p}}, \quad \forall \mathbf{q} \in \mathbb{S}_0(n).$$

Note that for  $i, j = 1, \dots, n-1$  and  $0 < \kappa < \frac{1}{2}$ , we have

$$\left( \int_{B^+(\bar{x}, \kappa r)} |D_{ij} v|^p \right)^{\frac{1}{p}} = \left( \int_{B^+(\bar{x}, \kappa r)} |D_{ij} v - D_{ij} v(\bar{x})|^p \right)^{\frac{1}{p}} \leq 2\kappa r \|D^3 v\|_{L^\infty(B(\bar{x}, \frac{1}{2}r))}.$$

Therefore, if we take  $\bar{\mathbf{q}}_{\bar{x}, \kappa r} \in \mathbb{S}_0(n)$  whose  $(i, n)$  entry is  $\overline{D_{in} v}_{B^+(\bar{x}, \kappa r)}$  for  $i = 1, \dots, n$ , then similar to (2.26), we have

$$\left( \int_{B^+(\bar{x}, \kappa r)} |D^2 v - \bar{\mathbf{q}}_{\bar{x}, \kappa r}|^p \right)^{\frac{1}{p}} \leq C_0 \kappa \left( \int_{B^+(\bar{x}, r)} |D^2 v - \mathbf{q}|^p \right)^{\frac{1}{p}}, \quad \forall \mathbf{q} \in \mathbb{S}_0(n).$$

By the same argument that led to (2.27), we find that there is  $\kappa \in (0, \frac{1}{2})$  such that

$$\varphi(\bar{x}, \kappa^j r) \leq 2^{-j} \varphi(\bar{x}, r) + C \|D^2 u\|_{L^\infty(B^+(\bar{x}, 2r))} \bar{\omega}_A(2\kappa^j r) + C \bar{\omega}_g(2\kappa^j r),$$

where  $\bar{\omega}_\bullet(t)$  is the same as in (2.28). The rest of proof is the same as that of Lemma 2.19.  $\blacksquare$

By modifying the proof of Lemmas 2.30, 2.41, and 2.46 in a straightforward way, we obtain the following lemmas.

**Lemma 2.52.** *For any  $x \in B_3^+$  and  $0 < \rho \leq r \leq \frac{1}{4}$ , we have*

$$\phi(x, \rho) \leq C \left( \frac{\rho}{r} \right)^\beta r^{-n} \|D^2 u\|_{L^1(B(x, 3r) \cap B_4^+)} + C \|D^2 u\|_{L^\infty(B(x, 5r) \cap B_4^+)} \hat{\omega}_A(\rho) + C \hat{\omega}_g(\rho),$$

where  $\beta \in (0, 1)$  and  $C > 0$  are constants depending only on  $n, \lambda, \Lambda$ , and  $p$ , and  $\hat{\omega}_\bullet(t)$  is a Dini function derived from  $\omega_\bullet(t)$ .

**Lemma 2.53.** *We have*

$$\|D^2 u\|_{L^\infty(B_2^+)} \leq C \|D^2 u\|_{L^1(B_4^+)} + C \int_0^1 \frac{\hat{\omega}_g(t)}{t} dt,$$

where  $C > 0$  is a constant depending only on  $n, \lambda, \Lambda, p$ , and  $\omega_A$ .

**Lemma 2.54.** *For any  $x \in B_3^+$  and  $0 < r \leq \frac{1}{5}$ , we have*

$$|D^2 u(x) - \mathbf{q}_{x,r}| \leq C r^\beta \|D^2 u\|_{L^1(B(x, \frac{3}{5}) \cap B_4^+)} + C \|D^2 u\|_{L^\infty(B(x, 1) \cap B_4^+)} \omega_A^*(r) + C \omega_g^*(r),$$

where  $\beta \in (0, 1)$  and  $C > 0$  are constants depending only on  $n, \lambda, \Lambda$ , and  $p$ , and  $\omega_\bullet^*(t)$  is defined as in (2.45).

With the above lemmas at hand, we obtain, similar to (2.48), the following estimates for  $x, y \in B_1^+$ :

$$\begin{aligned} |D^2 u(x) - D^2 u(y)| &\leq C \|D^2 u\|_{L^1(B_2^+)} |x - y|^\beta \\ &\quad + C \left( \|D^2 u\|_{L^1(B_4^+)} + \int_0^1 \frac{\hat{\omega}_g(t)}{t} dt \right) \omega_A^*(|x - y|) + C \omega_g^*(|x - y|), \end{aligned}$$

where  $\beta = \beta(n, \lambda, \Lambda, p)$  and  $C = C(n, \lambda, \Lambda, p, \omega_A)$ , and  $\omega_\bullet^*(t)$  is defined as in (2.45). We have shown that  $u \in C^2(\bar{B}_1^+)$  as desired. This completes the proof of Proposition 2.50 and that of Theorem 1.12.  $\blacksquare$

*Remark 2.55.* Instead of the condition of Dini mean oscillations (in the  $L^1$  sense), we may also consider coefficients and data with Dini mean oscillations in the  $L^p$  sense with some  $p \in (1, \infty)$ , i.e., the function  $\omega_{g,p} : \mathbb{R}_+ \rightarrow \mathbb{R}$  defined by

$$\omega_{g,p}(r) := \sup_{x \in \bar{\Omega}} \left( \int_{\Omega(x,r)} |g(y) - \bar{g}_{\Omega(x,r)}|^p dy \right)^{\frac{1}{p}}$$

is a Dini function. In this case, by modifying the proofs below and using the  $L^p$  estimates instead of the weak type-(1, 1) estimates, we can show that in the non-divergence case  $D^2 u$  has Dini mean oscillations in the  $L^p$  sense with the same  $p$ . Similar results hold for divergence form equations, and the adjoint problem of non-divergence form equations (with the boundary data  $\psi = 0$ ).

**2.4. Proof of Theorem 1.17.** As before, we adopt the usual summation convention over repeated indices. We first establish the following interior  $C^0$  estimates.

**Proposition 2.56.** *For any  $p \in (1, \infty)$ , we have  $u \in L^p(\Omega)$ . Moreover, for any  $\Omega' \subset\subset \Omega$ , we have  $u \in C(\overline{\Omega}')$ .*

*Proof.* Let  $w$  be a unique  $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$  solution of

$$\Delta w = D_i(b^i u) - cu + f \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega. \quad (2.57)$$

Since  $u \in L^2(\Omega)$ , we have  $w \in L^r(\Omega)$ , where  $\frac{1}{r} = \frac{1}{2} + \frac{1}{q} - \frac{1}{n} > \frac{1}{2}$ . Then, by setting  $\mathbf{g}' = \mathbf{g} + w\mathbf{I} \in L^r(\Omega)$ , we see that  $u$  becomes an adjoint solution of

$$D_{ij}(a^{ij}u) = \operatorname{div}^2 \mathbf{g}' \text{ in } \Omega, \quad u = \psi + \frac{\mathbf{g}' \nu \cdot \nu}{\mathbf{A} \nu \cdot \nu} \text{ on } \partial\Omega. \quad (2.58)$$

Therefore, by [9, Lemma 2], we see that  $u \in L^r(\Omega)$ . By bootstrapping, i.e., feeding  $u \in L^r(\Omega)$  back to (2.57), we find that  $u \in L^p(\Omega)$  for any  $p \in (1, \infty)$  with  $\|u\|_{L^p(\Omega)}$  controlled by the given data. Then, by the Morrey-Sobolev embedding, we have  $w \in C^{0,\mu}(\Omega)$  for some  $\mu > 0$  with  $\|w\|_{C^{0,\mu}(\Omega)}$  controlled by the given data, and

$$\varrho_w(t) \leq C \left( \|\mathbf{g}\|_{L^\infty(\Omega)} + \|f\|_{L^q(\Omega)} + \|\psi\|_{L^\infty(\partial\Omega)} + \|u\|_{L^1(\Omega)} \right) t^\mu.$$

Therefore,  $\omega_{\mathbf{g}'}$  is a Dini function that is completely determined by the given data. By [6, Theorem 1.10], we thus find that  $u \in C(\overline{\Omega}')$  and  $\|u\|_{C(\overline{\Omega}')}$  is bounded by a constant  $C$  depending only on the given data,  $\|u\|_{L^1(\Omega)}$ , and  $\Omega'$ . ■

Next, we turn to continuity estimate near the boundary. The following lemma shows that it is enough to consider the case when  $\psi = 0$  in (2.58).

**Lemma 2.59.** *The adjoint problem*

$$D_{ij}(a^{ij}v) = 0 \text{ in } \Omega, \quad v = \psi \text{ on } \partial\Omega$$

*has a unique solution  $v \in C(\overline{\Omega})$ .*

The proof of the above lemma is deferred to Section 3, where we introduce normalized adjoint solutions. Let  $\mathbf{g}'$  be as in the proof of Proposition 2.56. Under a mapping of flattening boundary

$$y = \Phi(x) = (\Phi^1(x), \dots, \Phi^n(x))$$

as before, let  $\tilde{u}(y) = u(x)$ , which satisfies

$$D_{ij}(\tilde{a}^{ij}\tilde{u}) = \operatorname{div}^2 \tilde{\mathbf{g}}' + D_i(\tilde{b}^i\tilde{u}),$$

where

$$\tilde{a}^{ij}(y) = D_l \Phi^i D_k \Phi^j a^{kl}(x), \quad \tilde{b}^i(y) = D_{kl} \Phi^i a^{kl}(x), \quad \tilde{\mathbf{g}}'(y) = D\Phi^T \mathbf{g}' D\Phi(x).$$

We may assume without loss of generality that  $\Phi$  is a  $C^{1,1}$ -diffeomorphism on  $\mathbb{R}^n$ . If we set  $\tilde{w}$  to be a solution to

$$\Delta \tilde{w} = D_i(\tilde{b}^i\tilde{u}) \text{ in } \Phi(\Omega), \quad \tilde{w} = 0 \text{ on } \Phi(\partial\Omega),$$

then  $\tilde{u}$  satisfies

$$D_{ij}(\tilde{a}^{ij}\tilde{u}) = \operatorname{div}^2(\tilde{\mathbf{g}}' + \tilde{w}\mathbf{I}) \text{ in } \Phi(\Omega), \quad \tilde{u} = \frac{(\tilde{\mathbf{g}}' + \tilde{w}\mathbf{I})\nu \cdot \nu}{\mathbf{A}\nu \cdot \nu} \text{ on } \Phi(\partial\Omega).$$

By Lemma 2.1 and Proposition 2.56, we see that the coefficients  $\tilde{a}^{ij}$  and the data  $\tilde{\mathbf{g}}' + \tilde{w}\mathbf{I}$  are of Dini mean oscillation. As before, we are thus reduced to prove the following.

**Proposition 2.60.** *If  $u \in L^2(B_4^+)$  is an adjoint solution satisfying*

$$D_{ij}(\tilde{a}^{ij}u) = \operatorname{div}^2 \mathbf{g} \text{ in } B_4^+, \quad u = \frac{\mathbf{g}^v \cdot v}{\mathbf{A}v \cdot v} \text{ on } T(0, 4),$$

*then  $u \in C(\overline{B}_1^+)$ .*

The rest of this subsection is devoted to the proof of Proposition 2.60. As in the proof of Propositions 2.14 and 2.50, we shall derive an a priori estimate of the modulus of continuity of  $u$  by assuming that  $u$  is in  $C(\overline{B}_3^+)$ .

Fix any  $p \in (0, 1)$ . Similar to (2.15) and (2.16), for  $x \in B_4^+$  and  $r > 0$ , we define

$$\phi(x, r) := \inf_{q \in \mathbb{R}} \left( \int_{B(x, r) \cap B_4^+} |u - q|^p \right)^{\frac{1}{p}}$$

and fix a number  $q_{x, r} \in \mathbb{R}$  satisfying

$$\phi(x, r) = \left( \int_{B(x, r) \cap B_4^+} |u - q_{x, r}|^p \right)^{\frac{1}{p}}.$$

The following lemma is in parallel with Lemmas 2.19 and 2.51.

**Lemma 2.61.** *For any  $\bar{x} \in T(0, 3)$  and  $0 < \rho \leq r \leq \frac{1}{2}$ , we have*

$$\phi(\bar{x}, \rho) \leq 2 \left( \frac{\rho}{r} \right)^\beta r^{-n} \|u\|_{L^1(B^+(\bar{x}, r))} + C \|u\|_{L^\infty(B^+(\bar{x}, 2r))} \tilde{\omega}_{\mathbf{A}}(2\rho) + C \tilde{\omega}_{\mathbf{g}}(2\rho),$$

where  $\beta \in (0, 1)$  and  $C > 0$  are constants depending only on  $n, \lambda, \Lambda$ , and  $p$ , and  $\tilde{\omega}_\bullet(t)$  is a Dini function derived from  $\omega_\bullet(t)$ .

*Proof.* For  $\bar{x} \in T(0, 3)$  and  $0 < r \leq \frac{1}{2}$ , we decompose  $u = v + w$ , where  $w \in L^2(B(\bar{x}, r))$  is a unique adjoint solution of the problem

$$\begin{aligned} D_{ij}(\tilde{a}^{ij}w) &= -\operatorname{div}^2((\mathbf{A} - \bar{\mathbf{A}})u) + \operatorname{div}^2(\mathbf{g} - \bar{\mathbf{g}}) \text{ in } \mathcal{D}(\bar{x}, 2r), \\ w &= \frac{(\mathbf{g} - \bar{\mathbf{g}} - (\mathbf{A} - \bar{\mathbf{A}})u) v \cdot v}{\bar{\mathbf{A}}v \cdot v} \text{ on } \partial\mathcal{D}(\bar{x}, 2r). \end{aligned}$$

By Lemma 2.5 with scaling, we have for any  $\alpha > 0$ ,

$$|\{x \in B^+(\bar{x}, r) : |w(x)| > \alpha\}| \lesssim \frac{1}{\alpha} \left( \|u\|_{L^\infty(B^+(\bar{x}, 2r))} \int_{B^+(\bar{x}, 2r)} |\mathbf{A} - \bar{\mathbf{A}}| + \int_{B^+(\bar{x}, 2r)} |\mathbf{g} - \bar{\mathbf{g}}| \right).$$

Therefore, we have

$$\left( \int_{B^+(\bar{x}, r)} |w|^p \right)^{\frac{1}{p}} \lesssim \omega_{\mathbf{A}}(2r) \|u\|_{L^\infty(B^+(\bar{x}, 2r))} + \omega_{\mathbf{g}}(2r).$$

Since  $v = u - w$  satisfies

$$D_{ij}(\tilde{a}^{ij}v) = \operatorname{div}^2 \bar{\mathbf{g}} \text{ in } B^+(\bar{x}, r), \quad v = \frac{\bar{\mathbf{g}}^v \cdot v}{\bar{\mathbf{A}}v \cdot v} \text{ on } T(\bar{x}, r),$$

by Lemma 2.7 with scaling, we have

$$\|Dv\|_{L^\infty(B^+(\bar{x}, \frac{1}{2}r))} \lesssim r^{-1} \left( \int_{B^+(\bar{x}, r)} |v - q|^p \right)^{\frac{1}{p}}, \quad \forall q \in \mathbb{R}.$$

Thus similar to (2.26), we have

$$\left( \int_{B^+(\bar{x}, \kappa r)} |v - \bar{v}_{\bar{x}, \kappa r}|^p \right)^{\frac{1}{p}} \leq C_0 \kappa \left( \int_{B^+(\bar{x}, r)} |v - q|^p \right)^{\frac{1}{p}}, \quad \forall q \in \mathbb{R}.$$

By the same argument that led to (2.27), we find that there is  $\kappa \in (0, \frac{1}{2})$  such that

$$\phi(\bar{x}, \kappa^j r) \leq 2^{-j} \phi(\bar{x}, r) + C \|u\|_{L^\infty(B^+(\bar{x}, 2r))} \tilde{\omega}_A(2\kappa^j r) + C \tilde{\omega}_g(2\kappa^j r),$$

where  $\tilde{\omega}_\bullet(t)$  is the same as in (2.28). The rest of proof is the same as that of Lemma 2.19.  $\blacksquare$

By modifying the proof of Lemmas 2.30, 2.41, and 2.46 in a straightforward way, we obtain the following lemmas.

**Lemma 2.62.** *For any  $x \in B_3^+$  and  $0 < \rho \leq r \leq \frac{1}{4}$ , we have*

$$\phi(x, \rho) \leq C \left( \frac{\rho}{r} \right)^\beta r^{-n} \|u\|_{L^1(B(x, 3r) \cap B_4^+)} + C \|u\|_{L^\infty(B(x, 5r) \cap B_4^+)} \hat{\omega}_A(\rho) + C \hat{\omega}_g(\rho),$$

where  $\beta \in (0, 1)$  and  $C > 0$  are constants depending only on  $n, \lambda, \Lambda$ , and  $p$ , and  $\hat{\omega}_\bullet(t)$  is a Dini function derived from  $\omega_\bullet(t)$ .

**Lemma 2.63.** *We have*

$$\|u\|_{L^\infty(B_2^+)} \leq C \|u\|_{L^1(B_4^+)} + C \int_0^1 \frac{\hat{\omega}_g(t)}{t} dt,$$

where  $C > 0$  is a constant depending only on  $n, \lambda, \Lambda, p$ , and  $\omega_A$ .

**Lemma 2.64.** *For any  $x \in B_3^+$  and  $0 < r \leq \frac{1}{5}$ , we have*

$$|u(x) - q_{x,r}| \leq C r^\beta \|u\|_{L^1(B(x, \frac{3}{5}) \cap B_4^+)} + C \|u\|_{L^\infty(B(x, 1) \cap B_4^+)} \omega_A^*(r) + C \omega_g^*(r),$$

where  $\beta \in (0, 1)$  and  $C > 0$  are constants depending only on  $n, \lambda, \Lambda$ , and  $p$ , and  $\omega_\bullet^*(t)$  is defined as in (2.45).

With the above lemmas at hand, we obtain, similar to (2.48), the following estimates for  $x, y \in B_1^+$ :

$$\begin{aligned} |u(x) - u(y)| &\leq C \|u\|_{L^1(B_2^+)} |x - y|^\beta \\ &\quad + C \left( \|u\|_{L^1(B_4^+)} + \int_0^1 \frac{\hat{\omega}_g(t)}{t} dt \right) \omega_A^*(|x - y|) + C \omega_g^*(|x - y|), \end{aligned}$$

where  $\beta = \beta(n, \lambda, \Lambda, p)$  and  $C = C(n, \lambda, \Lambda, p, \omega_A)$ , and  $\omega_\bullet^*(t)$  is defined as in (2.45). We have shown that  $u \in C(\bar{B}_1^+)$  as desired. This completes the proof of Proposition 2.60 and that of Theorem 1.17.  $\blacksquare$

## 3. WEAK TYPE-(1, 1) ESTIMATES

**3.1. Proof of Theorem 1.18.** We modify the proof of [6, Theorem 3.2]. Since the map  $T : f \mapsto Du$  is a bounded linear operator in  $L^2(\Omega)$ , it suffices to show that  $T$  satisfies the hypothesis of Lemma 4.1.

Set  $c = 8$ . For fixed  $x_0 \in \overline{\Omega}$  and  $0 < r < \frac{1}{2} \text{diam } \Omega$ , let  $\mathbf{b} \in L^2(\Omega)$  be supported in  $\Omega(x_0, r)$  and satisfy  $\int_{\Omega} \mathbf{b} = 0$ . Let  $u \in W_0^{1,2}(\Omega)$  be the unique weak solution of

$$\sum_{i,j=1}^n D_i(a^{ij}D_j u) = \text{div } \mathbf{b} \text{ in } \Omega; \quad u = 0 \text{ on } \partial\Omega.$$

For any  $R \geq 8r$  such that  $\Omega \setminus B(x_0, R) \neq \emptyset$  and  $\mathbf{g} \in C_c^\infty(\Omega(x_0, 2R) \setminus B(x_0, R))$ , let  $v \in W_0^{1,2}(\Omega)$  be a weak solution of an adjoint problem

$$\sum_{i,j=1}^n D_i(a^{ij}D_j v) = \text{div } \mathbf{g} \text{ in } \Omega; \quad v = 0 \text{ on } \partial\Omega.$$

Then, we have the identity

$$\int_{\Omega} Du \cdot \mathbf{g} = \int_{\Omega} \mathbf{b} \cdot Dv = \int_{\Omega(x_0, r)} \mathbf{b} \cdot (Dv - \overline{Dv}_{\Omega(x_0, r)}). \quad (3.1)$$

Since  $\mathbf{g} = 0$  in  $\Omega(x_0, R)$ , by flattening the boundary and using a similar argument that led to (2.48), we get

$$|Dv(x) - Dv(y)| \lesssim \left( \left( \frac{|x - y|}{R} \right)^\beta + \omega_{\mathbf{A}}^*(|x - y|) \right) R^{-\frac{d}{2}} \|Dv\|_{L^2(\Omega(x_0, \frac{1}{4}R))}$$

for  $x, y \in \Omega(x_0, \frac{1}{8}R)$ . Since  $r \leq R/8$ , we thus have

$$\|Dv - \overline{Dv}_{\Omega(x_0, r)}\|_{L^\infty(\Omega(x_0, r))} \lesssim R^{-\frac{d}{2}} \|Dv\|_{L^2(\Omega)} \left( r^\beta R^{-\beta} + \omega_{\mathbf{A}}^*(r) \right). \quad (3.2)$$

Using (1.19) and (2.45), it is routine to check that

$$\tilde{\omega}_{\mathbf{A}}(t) \lesssim (\ln r)^{-2}, \quad \hat{\omega}_{\mathbf{A}}(t) \lesssim (\ln r)^{-2}, \quad \omega_{\mathbf{A}}^*(t) \lesssim (\ln(4/r))^{-1}, \quad \forall r \in (0, \frac{1}{2}).$$

See, for instance, [6, Lemma 3.4]. By the construction of  $v$ , we have

$$\|Dv\|_{L^2(\Omega)} \lesssim \|\mathbf{g}\|_{L^2(\Omega)} = \|\mathbf{g}\|_{L^2(\Omega(x_0, 2R) \setminus B(x_0, R))}.$$

Therefore, we have by (3.1) and (3.2) that

$$\begin{aligned} \left| \int_{\Omega(x_0, 2R) \setminus B(x_0, R)} Du \cdot \mathbf{g} \right| &\lesssim \|\mathbf{b}\|_{L^1(\Omega(x_0, r))} \left( r^\beta R^{-\beta} + \{\ln(4/r)\}^{-1} \right) R^{-\frac{d}{2}} \|\mathbf{g}\|_{L^2(\Omega(x_0, 2R) \setminus B(x_0, R))}. \end{aligned}$$

and thus, by duality and Hölder's inequality, we get

$$\|Du\|_{L^1(\Omega(x_0, 2R) \setminus B(x_0, R))} \lesssim \left( r^\beta R^{-\beta} + \{\ln(4/r)\}^{-1} \right) \|\mathbf{b}\|_{L^1(\Omega(x_0, r))}.$$

Let  $N$  be the smallest positive integer such that  $\Omega \subset B(x_0, 2^{N+3}r)$ . Clearly,  $N \lesssim \ln(1/r)$ . By taking  $R = 2^3r, 2^4r, \dots, 2^{N+2}r$ , we have

$$\int_{\Omega \setminus B(x_0, 8r)} |Du| \lesssim \sum_{k=1}^N \left( 2^{-\beta k} + \{\ln(4/r)\}^{-1} \right) \|\mathbf{b}\|_{L^1(B(x_0, r))} \sim \int_{B(x_0, r)} |\mathbf{b}|,$$

and thus we are done.  $\blacksquare$

**3.2. Proof of Theorem 1.20.** For simplicity of argument, we may assume that  $\Omega$  is contained in  $B_5 = B(0, 5)$  and  $\mathbf{A}$  has Dini mean oscillation on  $B_{10} = B(0, 10)$ .

Let  $W$  be the adjoint solution to the problem

$$D_{ij}(a^{ij}W) = 0 \text{ in } B_{10}, \quad W = 1 \text{ on } \partial B_{10}.$$

It is known that  $W$  is a nonnegative Muckenhoupt weight in the reverse Hölder class  $\mathcal{B}_{\frac{n}{n-1}}(B_{10})$ , with constants which depend only on  $n$ ,  $\lambda$ , and  $\Lambda$ ; i.e.,

$$W(B(x_0, 2r)) \lesssim W(B(x_0, r)), \quad \left( W(B(x_0, r)) := \int_{B(x_0, r)} W dx \right), \quad (3.3)$$

$$\left( \int_{B(x_0, r)} W^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} \lesssim \int_{B(x_0, r)} W dx, \quad (3.4)$$

whenever  $B(x_0, 2r) \subset B_{10}$ ; see [10]. Also,  $W(B_{10}) \approx 1$ ; see [7, 11].

**Lemma 3.5.** For  $x_0 \in B_1$  and  $0 < r \leq 1$ , we have

$$\sup_{B(x_0, r)} W \leq C \int_{B(x_0, r)} W, \quad (3.6)$$

where  $C$  depends only on  $n$ ,  $\lambda$ ,  $\Lambda$ , and  $\omega_{\mathbf{A}}$ .

*Proof.* In the proof of [6, Theorem 1.10], it is shown that for  $x, y \in B(x_0, r)$

$$|W(x) - W(y)| \lesssim r^{-n} \|W\|_{L^1(B(x_0, 4r))} \left( r^{-\beta} |x - y|^\beta + \int_0^{|x-y|} \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t} dt \right),$$

where  $\tilde{\omega}_{\mathbf{A}}$  is as in (2.28). Therefore, we have

$$W(x) \lesssim r^{-n} \|W\|_{L^1(B(x_0, 4r))} \left( r^{-\beta} |x - y|^\beta + \int_0^{|x-y|} \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t} dt \right) + W(y).$$

By averaging over  $y \in B(x_0, r)$ ,

$$W(x) \lesssim \left( 1 + \int_0^{2r} \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t} dt \right) \int_{B(x_0, 4r)} W + \int_{B(x_0, r)} W.$$

By using the doubling property (3.3) of  $W$ , we get (3.6).  $\blacksquare$

**Definition 3.7.** We say that  $\tilde{v}$  is a normalized adjoint solution (for the operator  $a^{ij}D_{ij}$ ) in an open subset  $\Omega'$  of  $\Omega$  if  $\tilde{v}$  is a continuous function defined in  $\Omega'$  such that  $\tilde{v}W$  is an adjoint solution in  $\Omega'$ , i.e.,  $D_{ij}(a^{ij}\tilde{v}W) = 0$  in  $\Omega'$ .

We record the following property of normalized adjoint solutions  $\tilde{v}$  on  $B(x_0, r) \subset \Omega$ : There are constants depending only on  $\lambda$ ,  $\Lambda$ , and  $n$  such that the following holds:

$$\|\tilde{v}\|_{L^\infty(B(x_0, \frac{r}{2}))} \leq \frac{C}{W(B(x_0, r))} \int_{B(x_0, r)} |\tilde{v}|W dx, \quad (3.8)$$

$$[\tilde{v}]_{C^\alpha(B(x_0, r))} \leq Cr^{-\alpha} \|\tilde{v}\|_{L^\infty(B(x_0, 2r))}.$$

See [1, 7]. There is also a boundary version of the above estimates. Namely, if  $\tilde{v}$  is a normalized adjoint solution in  $\Omega(x_0, r)$  with  $x_0 \in \overline{\Omega}$  and  $\tilde{v} = 0$  on  $B(x_0, r) \cap \partial\Omega$ , then

$$\|\tilde{v}\|_{L^\infty(\Omega(x_0, \frac{r}{2}))} \leq \frac{C}{W(B(x_0, r))} \int_{\Omega(x_0, r)} |\tilde{v}|W. \quad (3.9)$$



and

$$[\tilde{v}]_{C^\alpha(B(x_0, r))} \leq Cr^{-\alpha} \|\tilde{v}\|_{L^\infty(\Omega(x_0, 2r))}. \quad (3.10)$$

We note that the constants  $C$  and  $0 < \alpha \leq 1$  in the above estimates depend only on  $n$ ,  $\lambda$ , and  $\Lambda$ ; see [1, 11].

We are now ready to prove the theorem. We shall make first the qualitative assumption that the coefficients  $\mathbf{A} = (a^{ij})$  are smooth. However, the constant  $C$  that appears in (1.22) will not depend on the extra smoothness of the coefficients. Let  $\{Q_l\}$  be a collection of disjoint “cubes” as those used in the proof of Lemma 4.1 so that we have

$$t < \int_{Q_l} |f| \leq A_1 t \quad (3.11)$$

and  $|f(x)| \leq t$  for a.e.  $x \in \Omega \setminus \bigcup_l Q_l$ . We decompose  $f = g + b$ , with  $b = \sum_l b_l$ , such that

$$g = \tilde{m}_l(f) := \frac{1}{W(Q_l)} \int_{Q_l} f W \text{ on } Q_l,$$

$g = f$  on  $\Omega \setminus \bigcup_l Q_l$ , and set

$$b_l = \chi_{Q_l} (f - \tilde{m}_l(f)).$$

It is obvious that

$$\int_{\Omega} b_l W = \int_{Q_l} b_l W = 0. \quad (3.12)$$

By (3.6) we find (a ball  $B(x_0, r)$  can be easily replaced by a “cube”)

$$|\tilde{m}_l(f)| \leq \frac{1}{W(Q_l)} \|W\|_{L^\infty(Q_l)} \int_{Q_l} |f| \lesssim \int_{Q_l} |f|$$

and thus, we have

$$\int_{Q_l} |b_l| dx \leq \int_{Q_l} |f| + |Q_l| |\tilde{m}_l(f)| \lesssim \int_{Q_l} |f| \lesssim t |Q_l|. \quad (3.13)$$

Also, we find that (c.f. (4.6) in Appendix)

$$|g(x)| \lesssim t, \quad \text{for a.e. } x \in \Omega.$$

Now, we write  $u = v + w$ , where  $v \in W_0^{1,2}(B_1) \cap W^{2,2}(B_1)$  is a unique solution to

$$a^{ij} D_{ij} v = g \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega.$$

By the standard elliptic theory, we have

$$\|D^2 v\|_{L^2(\Omega)} \lesssim \|g\|_{L^2(\Omega)}$$

and thus, we have (c.f. (4.7) in Appendix)

$$\begin{aligned} \left| \{x \in \Omega : |D^2 v(x)| > \tfrac{1}{2}t\} \right| &\lesssim \frac{1}{t^2} \int_{\Omega} |D^2 v|^2 \lesssim \frac{1}{t^2} \int_{\Omega} |g|^2 \\ &\lesssim \frac{1}{t} \int_{\Omega \setminus \bigcup_l Q_l} |f| + \sum_l |Q_l| \lesssim \frac{1}{t} \int_{\Omega} |f|. \end{aligned} \quad (3.14)$$

For each  $l = 1, 2, \dots$ , let  $w_l \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$  be the unique solution to

$$a^{ij} D_{ij} w_l = b_l \text{ in } \Omega, \quad w_l = 0 \text{ on } \partial\Omega. \quad (3.15)$$

Take  $c = 8$ . We associate each  $Q_l$  with a ball  $B_l = B(x_l, r_l)$  as in the proof of Lemma 4.1, and denote  $B_l^* = B(x_l, 8r_l)$ . Since  $w = \sum_l w_l$ , we have

$$\int_{\Omega \setminus \bigcup_l B_l^*} |D^2 w| \leq \sum_l \int_{\Omega \setminus B_l^*} |D^2 w_l|.$$

We claim that

$$\int_{\Omega \setminus B_l^*} |D^2 w_l| \lesssim \int_{Q_l} |b_l|. \quad (3.16)$$

Take the claim for now. Then, by (3.13) and (3.11), we get

$$\int_{\Omega \setminus \bigcup_l B_l^*} |D^2 w| \lesssim \sum_l \int_{Q_l} |b_l| \lesssim t \sum_l |Q_l| \lesssim \int_{\Omega} |f|,$$

which shows that

$$|\{x \in \Omega : |D^2 w(x)| > \tfrac{1}{2}t\} \setminus \bigcup_l Q_l^*| \lesssim \frac{1}{t} \int_{\Omega} |f|.$$

However,

$$|\bigcup_l B_l^*| \leq 8^n \sum_l |B_l| \lesssim \sum_l |Q_l| \lesssim \frac{1}{t} \int_{\Omega} |f|.$$

Together then, the last two estimates imply

$$|\{x \in \Omega : |D^2 w(x)| > \tfrac{1}{2}t\}| \lesssim \frac{1}{t} \int_{\Omega} |f|,$$

which combined with (3.14) gives the theorem since  $u = v + w$ .

We now prove the claim (3.16). To do this, we follow the same line of proof of [6, Lemma 2.20]. Recall that  $w_l \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$  satisfies (3.15) with  $b_l$  supported in  $Q_l \subset B_l \cap \Omega = \Omega(x_l, r_l)$ . For any  $R \geq 8r_l$  such that  $\Omega \setminus B(x_l, R) \neq \emptyset$  and smooth functions  $\mathbf{h} = (h^{km})_{k,m=1}^n$  with a compact support in  $\Omega(x_l, 2R) \setminus \overline{B}(x_l, R)$ , let  $v_l \in L^2(\Omega)$  be a unique adjoint solution of

$$D_{ij}(a^{ij}v_l) = \operatorname{div}^2 \mathbf{h} \text{ in } \Omega; \quad v_l = 0 \text{ on } \partial\Omega,$$

and let

$$\tilde{v}_l := v_l / W.$$

Then, we have the identity

$$\int_{\Omega} \operatorname{tr}(D^2 w_l \mathbf{h}) = \int_{\Omega} v_l b_l = \int_{\Omega} \tilde{v}_l W b_l = \int_{Q_l} (\tilde{v}_l - (\tilde{v}_l)_{x_l, r_l}) W b_l, \quad (3.17)$$

where we set

$$(\tilde{v}_l)_{x_l, r_l} := \oint_{\Omega(x_l, r_l)} \tilde{v}_l$$

and used (3.12). Since  $\mathbf{h} = 0$  in  $\Omega(x_l, R)$ , we find that  $\tilde{v}_l$  is a normalized adjoint solution in  $\Omega(x_l, R)$ . Thus, by (3.9) and (3.10), for  $x \in Q_l$ , we have

$$\begin{aligned} |\tilde{v}_l(x) - (\tilde{v}_l)_{x_l, r_l}| &\leq \oint_{\Omega(x_l, r_l)} |\tilde{v}_l(x) - \tilde{v}_l(y)| dy \lesssim \left(\frac{r_l}{R}\right)^\alpha \|\tilde{v}_l\|_{L^\infty(\Omega(x_l, \frac{1}{4}R))} \\ &\lesssim \left(\frac{r_l}{R}\right)^\alpha \frac{1}{W(B(x_l, \frac{1}{2}R))} \int_{\Omega(x_l, R)} |\tilde{v}_l| W = \left(\frac{r_l}{R}\right)^\alpha \frac{1}{W(B(x_l, \frac{1}{2}R))} \int_{\Omega(x_l, R)} |v_l|. \end{aligned}$$

Therefore, by (3.17), (3.6), and the doubling property of  $W$ , we have

$$\begin{aligned}
\left| \int_{\Omega(x_l, 2R) \setminus B(x_l, R)} \text{tr}(D^2 w_l \mathbf{h}) \right| &\lesssim \left( \frac{r_l}{R} \right)^\alpha \frac{1}{W(B(x_l, \frac{1}{2}R))} \|v_l\|_{L^1(\Omega(x_l, R))} \|W\|_{L^\infty(B(x_l, \frac{1}{4}R))} \|b_l\|_{L^1(Q_l)} \\
&\lesssim \left( \frac{r_l}{R} \right)^\alpha \frac{W(B(x_l, \frac{1}{4}R))}{W(B(x_l, \frac{1}{2}R)) |B(x_l, \frac{1}{4}R)|} \|v_l\|_{L^1(\Omega(x_l, R))} \|b_l\|_{L^1(Q_l)} \\
&\lesssim r_l^\alpha R^{-\alpha - \frac{n}{2}} \|b_l\|_{L^1(Q_l)} \|v_l\|_{L^2(\Omega)} \\
&\lesssim r_l^\alpha R^{-\alpha - \frac{n}{2}} \|b_l\|_{L^1(Q_l)} \|\mathbf{h}\|_{L^2(\Omega(x_l, 2R) \setminus B(x_l, R))},
\end{aligned}$$

where we used Hölder's inequality and the estimate

$$\|v_l\|_{L^2(\Omega)} \lesssim \|\mathbf{h}\|_{L^2(\Omega)}$$

in the last two inequalities. Therefore, by duality and Hölder's inequality, we get

$$\|D^2 w_l\|_{L^1(\Omega(x_l, 2R) \setminus B(x_l, R))} \lesssim r_l^\alpha R^{-\alpha} \|b_l\|_{L^1(Q_l)}.$$

Now let  $N$  be the smallest positive integer such that  $\Omega \subset B(x_l, 2^N 8r_l)$ . By taking  $R = 2^3 r_l, 2^4 r_l, \dots, 2^{N+2} r_l$  in the above, we have

$$\int_{\Omega \setminus B(x_l, 8r_l)} |D^2 w_l| \lesssim \sum_{k=1}^N 2^{-\alpha k} \|b_l\|_{L^1(Q_l)} \approx \int_{Q_l} |b_l|$$

as desired.

Finally, we shall show how to get rid of extra smoothness assumption on the coefficients  $\mathbf{A} = (a^{ij})$ . Let  $\mathbf{A}_k = (a_k^{ij})$  and  $f_k$  are smooth functions such that  $a_k^{ij}$  converges uniformly to  $a^{ij}$  in  $\Omega$  and  $f_k$  converges to  $f$  in  $L^2(\Omega)$  as  $k \rightarrow \infty$ . We may further assume that  $\mathbf{A}_k$  satisfies condition (1.10) and that  $\omega_{\mathbf{A}_k} \leq \omega_{\mathbf{A}}$ . Let  $u_k \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$  satisfy

$$a_k^{ij} D_{ij} u_k = f_k \text{ in } \Omega, \quad u_k = 0 \text{ on } \partial\Omega.$$

Then, by (1.22), we have an estimate

$$|\{x \in \Omega : |D^2 u_k(x)| > t\}| \leq \frac{C}{t} \int_{\Omega} |f_k| dx, \quad (3.18)$$

which is uniform for all  $k = 1, 2, \dots$ . Note that

$$a_k^{ij} D_{ij}(u - u_k) = f - f_k + (a_k^{ij} - a^{ij}) D_{ij} u$$

so that, by the  $W^{2,2}$  theory, we have

$$\|D^2 u - D^2 u_k\|_{L^2(\Omega)} \leq C \left( \|f - f_k\|_{L^2(\Omega)} + \|(a_k^{ij} - a^{ij}) D_{ij} u\|_{L^2(\Omega)} \right).$$

In particular, we find that  $D^2 u_k \rightarrow D^2 u$  in  $L^2(\Omega)$ . Therefore, by taking limit  $k \rightarrow \infty$  in (3.18), we get (1.22).  $\blacksquare$

*Remark 3.19.* It is shown in [7] that when the coefficients of the elliptic operator  $\mathcal{L}$  in non-divergence form are VMO functions in  $B_{10}$ , i.e.,

$$\lim_{r \rightarrow 0^+} \sup_{x \in B_{10}} \int_{Q(x,r)} |\mathbf{A} - \bar{\mathbf{A}}_{Q(x,r)}| = 0, \quad (3.20)$$

where  $Q(x, r)$  denotes a cube in  $\mathbb{R}^n$  centered at  $x$  with edges of length  $r$  and sides parallel to the coordinate axis, the Muckenhoupt property (3.4) of the weight  $W$  improves because then,  $\log W$  is in  $\text{VMO}(B_5)$  (see [7, Theorem 1.2]), i.e.,

$$\lim_{r \rightarrow 0^+} \sup_{x \in B_5} \int_{Q(x, r)} |\log W - \overline{\log W}_{Q(x, r)}| = 0, \quad (3.21)$$

while the weight  $W$  is shown to be unbounded or to vanish inside  $B_{10}$  for some of these operators; see [7, §3]. In fact, when  $\mathbf{A}$  has Dini mean oscillation in  $B_{10}$ , the ideas behind [6, Theorem 1.10] and [9, Lemma 3] imply that nonnegative adjoint solutions  $W$  verify the following Harnack inequality: there is  $C = C(\lambda, \Lambda, n, \omega_{\mathbf{A}})$  such that

$$\sup_{B(x, r)} W \leq C \inf_{B(x, r)} W,$$

when  $B(x, 2r) \subset B_8$ . See Lemma 4.9 in Appendix. This local Harnack inequality fails when  $\mathbf{A}$  is continuous on  $B_{10}$ ; see [7, §3].

It is shown in [7, Theorem 1.3] that under the same hypothesis, the solution  $u$  to (1.21) verifies an interior weak type-(1, 1) property with respect to the weight  $W dx$ , when  $f \in C_c^\infty(\Omega)$ . Also, [7, §3] provides counterexamples of operators  $\mathcal{L}$  in non-divergence form with continuous coefficients for which the weak type-(1, 1) (1.22) fails in the interior of  $\Omega$ . Notice that the coefficient matrices there just fail to be Dini continuous or have Dini mean oscillations.

Finally, the methods in this paper, (3.3), (3.8) – (3.10) and (3.21) are easily seen to help to extend up to the boundary the interior weak type-(1, 1) property with respect to the weight  $W dx$  in [7, Theorem 1.3]. In particular, under the weaker VMO condition (3.20) one can show that there is  $C = C(n, \lambda, \Lambda, \Omega, \mathbf{A})$  such that for any  $t > 0$ ,

$$W(\{x \in \Omega : |D^2 u(x)| > t\}) \leq \frac{C}{t} \int_{\Omega} |f| W dx,$$

when  $u$  is the solution to (1.21),  $\Omega \subset B_5$ , and  $f \in C^\infty(\overline{\Omega})$ . Observe there are operators with uniformly continuous coefficients such that the adjoint solution  $W$  is unbounded above or below or it is not a local  $A_1$  Muckenhoupt weight (See [2] and [7, §3]).

**3.3. Proof of Lemma 2.59.** As in the proof of Theorem 1.20, let us assume that  $\Omega$  is contained in  $B_5 = B(0, 5)$  and  $\mathbf{A}$  has Dini mean oscillation on  $B_{10}$ . Let  $W$  be as given in the proof of Theorem 1.20. By [6, Theorem 1.10], we find that  $W$  is uniformly continuous in  $B_5$  with its modulus of continuity determined by  $n, \lambda, \Lambda$ , and  $\omega_{\mathbf{A}}$ . Also, Lemma 4.9 in Appendix implies that  $W$  is bounded from above and below in  $B_5$  with its lower and upper bounds depending only on  $n, \lambda, \Lambda$ , and  $\omega_{\mathbf{A}}$ .

Therefore, by [8, Theorem 2.8], there is a unique normalized adjoint solution  $\tilde{v}$  that satisfies

$$D_{ij}(a^{ij} \tilde{v} W) = 0 \text{ in } \Omega, \quad \tilde{v} = \frac{\psi}{W} \text{ on } \partial\Omega.$$

Moreover,  $\tilde{v} \in C(\overline{\Omega})$  with a modulus of continuity controlled by  $n, \lambda, \Lambda$ , the Lipschitz character of  $\Omega$ , and the modulus of continuity of  $\frac{\psi}{W}$ . The latter in turn is controlled by  $n, \lambda, \Lambda, \omega_{\mathbf{A}}$ , and  $\psi$ . It is clear that  $v = \tilde{v} W$  satisfies all the desired properties. ■

## 4. APPENDIX

The following lemma is a slight generalization of [6, Lemma 2.1]. For the completeness, we present a proof here.

**Lemma 4.1.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain satisfying the condition (1.5) and let  $T$  be a bounded linear operator from  $L^2(\Omega)$  to  $L^2(\Omega)$ . Let  $\mu \in (0, 1)$  be a constant. Suppose that for any  $x_0 \in \Omega$  and  $0 < r < \mu \text{diam } \Omega$ , we have*

$$\int_{\Omega \setminus B(x_0, cr)} |Tb| \leq C \int_{B(x_0, r) \cap \Omega} |b| \quad (4.2)$$

whenever  $b \in L^2(\Omega)$  is supported in  $B(x_0, r) \cap \Omega$ ,  $\int_{\Omega} b = 0$ , and  $c > 1$  and  $C > 0$  are constants. Then for  $f \in L^2(\Omega)$  and any  $t > 0$ , we have

$$|\{x \in \Omega : |Tf(x)| > t\}| \leq \frac{C'}{t} \int_{\Omega} |f|, \quad (4.3)$$

where  $C' = C'(n, c, C, \mu, \Omega, A_0)$  is a constant.

*Proof.* To begin with, we note that  $\Omega$  equipped with the standard Euclidean metric and the Lebesgue measure (restricted to  $\Omega$ ) is a space of homogeneous type. By [4, Theorem 11], there exists a collection of open subsets (called “cubes”)

$$\{Q_{\alpha}^k \subset \Omega : k \in \mathbb{Z}, \alpha \in I_k\},$$

with  $I_k$  at most countable set and constants  $\delta \in (0, 1)$ ,  $a_0 > 0$  and  $C_1 < \infty$  such that

- i)  $|\Omega \setminus \bigcup_{\alpha} Q_{\alpha}^k| = 0 \quad \forall k$ .
- ii) If  $\ell \geq k$  then either  $Q_{\beta}^{\ell} \subset Q_{\alpha}^k$  or  $Q_{\beta}^{\ell} \cap Q_{\alpha}^k = \emptyset$ .
- iii) For each  $(k, \alpha)$  and each  $\ell < k$  there is a unique  $\beta$  such that  $Q_{\alpha}^k \subset Q_{\beta}^{\ell}$ .
- iv)  $\text{diam } Q_{\alpha}^k \leq C_1 \delta^k$ .
- v) Each  $Q_{\alpha}^k$  contains some “ball”  $B(z_{\alpha}^k, a_0 \delta^k) \cap \Omega$ .

From the above, we can infer the following.

- (a) There is constant  $A_1 \geq 1$  such that if  $Q_{\beta}^{k-1}$  is the parent of  $Q_{\alpha}^k$  (resp. if  $B_{\alpha}^k$  is the Euclidean ball in  $\mathbb{R}^n$  centered at  $z_{\alpha}^k$  with radius  $r = \text{diam } Q_{\alpha}^k$ ), then we have

$$|Q_{\beta}^{k-1}| \leq A_1 |Q_{\alpha}^k| \quad (\text{resp. } |B_{\alpha}^k| \leq A_1 |Q_{\alpha}^k|). \quad (4.4)$$

- (b) The Lebesgue differentiation theorem is available for the chain of cubes shrinking to a point because the maximal function defined as

$$M(f)(x) = \begin{cases} \sup_{x \in Q_{\alpha}^k} \int_{Q_{\alpha}^k} |f| dx, & \text{when } x \in \bigcap_k \bigcup_{\alpha \in I_k} Q_{\alpha}^k, \\ 0, & \text{otherwise,} \end{cases}$$

is of weak type-(1, 1) over  $\Omega$ .

By i) – v) above and (1.5), choose  $k_0 \in \mathbb{Z}$  with  $\theta = \inf_{\alpha \in I_{k_0}} |Q_{\alpha}^{k_0}| > 0$ . To get (4.3) when

$$\frac{1}{t} \int_{\Omega} |f| dx > \theta,$$

it suffices to choose  $C' \geq \theta^{-1} |\Omega|$ . Otherwise,

$$\int_{Q_{\alpha}^{k_0}} |f| dx \leq t, \text{ for all } \alpha \in I_{k_0}.$$

Let then  $\{Q_l\}$  denote the set of cubes chosen as follows. For  $k = k_0 + 1$  and  $\alpha \in I_k$ , the cube  $Q = Q_\alpha^k$  satisfies either  $\int_Q |f| \leq t$  or  $\int_Q |f| > t$ . In the second case, we select  $Q = Q_\alpha^k$  as one of the cubes in  $\{Q_l\}$ . Note that in this case, we have by (4.4)

$$t < \int_Q |f| dx \leq A_1 t.$$

In the first case, we subdivide  $Q = Q_\alpha^k$  further into subcubes  $Q' = Q_\beta^{k+1}$ , and repeat the process until (if ever) we are forced into the second case. By observation (b), we find that  $|f(x)| \leq t$  for a.e.  $x \in \Omega \setminus \bigcup_l Q_l$ .

We decompose  $f = g + b$ , with  $b = \sum_l b_l$ , such that

$$g = m_l(f) := \int_{Q_l} f dx \text{ on } Q_l,$$

$g = f$  on  $\Omega \setminus \bigcup_l Q_l$ , and set

$$b_l = \chi_{Q_l}(f - m_l(f)).$$

It is obvious that  $\int_{Q_l} b_l dx = 0$  and we have

$$\int_{Q_l} |b_l| dx \leq \int_{Q_l} |f| dx + |Q_l| m_l(f) \leq 2 \int_{Q_l} |f| dx \leq 2A_1 t |Q_l|. \quad (4.5)$$

Also, we see that

$$|g(x)| \leq A_1 t \text{ for a.e. } x \in \Omega. \quad (4.6)$$

Indeed, for a.e.  $x \in \Omega \setminus \bigcup_l Q_l$ , we have  $|g(x)| = |f(x)| \leq t$  and  $|g(x)| \leq A_1 t$  on  $Q_l$ . By Chebyshev's inequality and the  $L^2$  boundedness of  $T$ , we have

$$\begin{aligned} \left| \{x \in \Omega : |Tg(x)| > \tfrac{1}{2}t\} \right| &\lesssim \frac{1}{t^2} \int_{\Omega} |Tg|^2 dx \lesssim \frac{1}{t^2} \int_{\Omega} |g|^2 dx \\ &\lesssim \frac{1}{t} \int_{\Omega \setminus \bigcup_l Q_l} |f| dx + \sum_l |Q_l| \lesssim \frac{1}{t} \int_{\Omega} |f| dx, \end{aligned} \quad (4.7)$$

where we used (4.6) and the property that

$$\sum_l |Q_l| \leq \frac{1}{t} \int_{\Omega} |f| dx. \quad (4.8)$$

We associate each  $Q_l = Q_\alpha^k$  with a Euclidean ball  $B_l = B(x_l, r_l)$ , where  $x_l = z_\alpha^k \in \Omega$  and  $r_l = \text{diam } Q_\alpha^k$ . Let us denote  $B_l^* = B(x_l, 8r_l)$ . Since  $Tb = \sum_l Tb_l$ , we have

$$\int_{\Omega \setminus \bigcup_l B_l^*} |Tb| dx \leq \sum_l \int_{\Omega \setminus B_l^*} |Tb_l| dx.$$

By the hypothesis (4.2) together with (4.5) and (4.8), we get

$$\int_{\Omega \setminus \bigcup_l B_l^*} |Tb| dx \leq C \sum_l \int_{Q_l} |b_l| dx \lesssim t \sum_l |Q_l| \lesssim \int_{\Omega} |f| dx,$$

which via Chebyshev's inequality shows that

$$\left| \{x \in \Omega : |Tb(x)| > \tfrac{1}{2}t\} \setminus \bigcup_l B_l^* \right| \lesssim \frac{1}{t} \int_{\Omega} |f| dx.$$

Also, by (4.4), we have

$$|\cup_l B_l^*| \leq 8^n \sum_l |B_l| \leq 8^n A_1 \sum_l |Q_l| \lesssim \frac{1}{t} \int_{\Omega} |f| dx.$$

Together then, the last two estimates imply

$$|\{x \in \Omega : |Tb(x)| > \frac{1}{2}t\}| \lesssim \frac{1}{t} \int_{\Omega} |f| dx,$$

which combined with (4.7) gives (4.3) since  $Tf = Tg + Tb$ .  $\blacksquare$

Finally we prove the following Harnack type inequality for nonnegative adjoint solutions.

**Lemma 4.9.** *Assume the coefficients  $\mathbf{A} = (a^{ij})$  are of Dini mean oscillation and satisfy the condition (1.10). Let  $w \in L^2(B_4)$  be a nonnegative solution to  $D_{ij}(a^{ij}w) = 0$  in  $B_4 = B(0, 4)$  and  $\|w\|_{L^1(B_3)} = 1$ . Then we have*

$$c \leq \inf_{B_1} w, \quad \sup_{B_1} w \leq C,$$

where  $c$  and  $C$  are positive constants depending only on  $n, \lambda, \Lambda$ , and  $\omega_{\mathbf{A}}$ .

*Proof.* The upper bound follows from Lemma 3.5. In particular, it follows from [6, (2.25)] that for any  $y_0 \in B_1$  and  $R \in (0, 1]$ , we have

$$|w(x) - w(y_0)| \leq C \left( \left( \frac{|x - y_0|}{R} \right)^{\beta} + \int_0^{|x - y_0|} \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t} dt \right) R^{-d} \|w\|_{L^1(B(y_0, R))} \quad (4.10)$$

for  $x \in B(y_0, \frac{1}{2}R)$ . Here  $\beta > 0$  is an absolute constant and  $\tilde{\omega}_{\mathbf{A}}$  is defined as in (2.28).

We prove the lower bound by contradiction. Suppose the claim is not true. Then we can find a sequence of coefficients  $\{\mathbf{A}_k\}$  satisfying

$$\sup_k \omega_{\mathbf{A}_k}(t) \leq \omega(t)$$

for some Dini function  $\omega$  and a sequence of nonnegative solutions  $\{w_k\}$  to

$$D_{ij}(a_k^{ij}w_k) = 0 \quad \text{in } B_4$$

such that

$$\|w_k\|_{L^1(B_3)} = 1 \quad \text{and} \quad w_k(x_k) \leq 1/k$$

for some  $x_k \in B_1$ . After passing to a subsequence, we may assume that  $x_k \rightarrow y_0 \in \bar{B}_1$ . By [6, Theorem 1.10],  $\{w_k\}$  is uniformly bounded and equicontinuous in  $\bar{B}_2$ . Of course,  $\{\mathbf{A}_k\}$  is also uniformly bounded and equicontinuous in  $\bar{B}_2$ . Therefore, by the Arzelà–Ascoli theorem, they have subsequences, still denoted by  $\{w_k\}$  and  $\{\mathbf{A}_k\}$ , which converge to  $w$  and  $\mathbf{A}$  uniformly in  $\bar{B}_2$ , with the same moduli of continuity. It is easily seen that  $w$  is a nonnegative solution of

$$D_{ij}(a^{ij}w) = 0 \quad \text{in } B_2$$

and  $w(y_0) = 0$ . By the doubling property of  $w$  (see [10]),  $\|w_k\|_{L^1(B_2)}$  is bounded from below and above uniformly. It then follow from the uniform convergence that  $\|w\|_{L^1(B_2)}$  is also bounded from below and above.

Let  $\kappa \in (0, 1/2)$  be a small constant to be specified later. From (4.10), for any  $R \in (0, 1]$ , we have

$$\oint_{B(y_0, \kappa R)} w \leq N \left( \kappa^\beta + \int_0^{\kappa R} \frac{\tilde{\omega}_A(t)}{t} dt \right) \oint_{B(y_0, R)} w,$$

where  $N$  is independent of  $\kappa$ . We then fix  $\kappa$  sufficiently small such that  $2N\kappa^\beta \leq \kappa^{\beta/2}$ . Then for any small  $R$  such that

$$\int_0^{\kappa R} \frac{\tilde{\omega}_A(t)}{t} dt \leq \kappa^\beta,$$

we obtain

$$\oint_{B(y_0, \kappa R)} w \leq \kappa^{\beta/2} \oint_{B(y_0, R)} w.$$

By iteration, we deduce  $\oint_{B(y_0, r)} w \leq Nr^{\beta/2}$ . This, however, contradicts with [8, Theorem 1.5], which reads that for any  $\varepsilon > 0$ , it holds that  $\oint_{B(y_0, r)} w \gtrsim r^\varepsilon$  for all  $r \in (0, 1)$ . ■

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(H. Dong) DIVISION OF APPLIED MATHEMATICS, BROWN UNIVERSITY, 182 GEORGE STREET, PROVIDENCE, RI 02912, UNITED STATES OF AMERICA  
*E-mail address*: Hongjie.Dong@brown.edu

(L. Escauriaza) UPV/EHU, DPTO. MATEMÁTICAS, BARRIO SARRIENA s/N 48940 LEIOA, SPAIN  
*E-mail address*: luis.escauriaza@ehu.eus

(S. Kim) DEPARTMENT OF MATHEMATICS, YONSEI UNIVERSITY, 50 YONSEI-RO, SEODAEMUN-GU, SEOUL 03722, REPUBLIC OF KOREA  
*E-mail address*: kimseick@yonsei.ac.kr